Bubbles and Manias*

(Preliminary version)

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Abstract

This paper presents an asset-trading model of homogeneous information with rational and behavioral agents. We provide some conditions for the existence of an asset pricing bubble. Broadly speaking, a bubble arises if and only if there is a positive probability of a mania: a subset of states in which the absorbing capacity of the behavioral agents is greater than the maximal selling pressure of the rational agents.

JEL: G12, G14.

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1. Introduction

When we see a stock price path like the one depicted in Figure 1 an all too familiar kind of thought usually comes to mind: If only I. . . (had bought stocks in 1995 and sold them in 2000). Such feelings of regret may be inevitable to some but do not stay for long: Even if the first thing that comes to mind is the lost opportunity of earning over 150% in five years, the second, consoling thing we tell ourselves upon reflection is that nobody could know in advance when to buy and sell. Indeed, we further know this second thought is well-grounded in reason: There is no such thing as free lunch and the 1995–2003 run-up and collapse we see in Figure 1 could take place only because it was unpredictable.

Narrative accounts of dramatic asset price increases followed by a collapse usually go as follows (Galbraith, 1994; Shiller, 2000a; Kindleberger and Aliber, 2005). After some particularly good news about the profitability of a certain investment, smart investors buy assets bidding up their price. The initial price rise calls the attention of outsiders who

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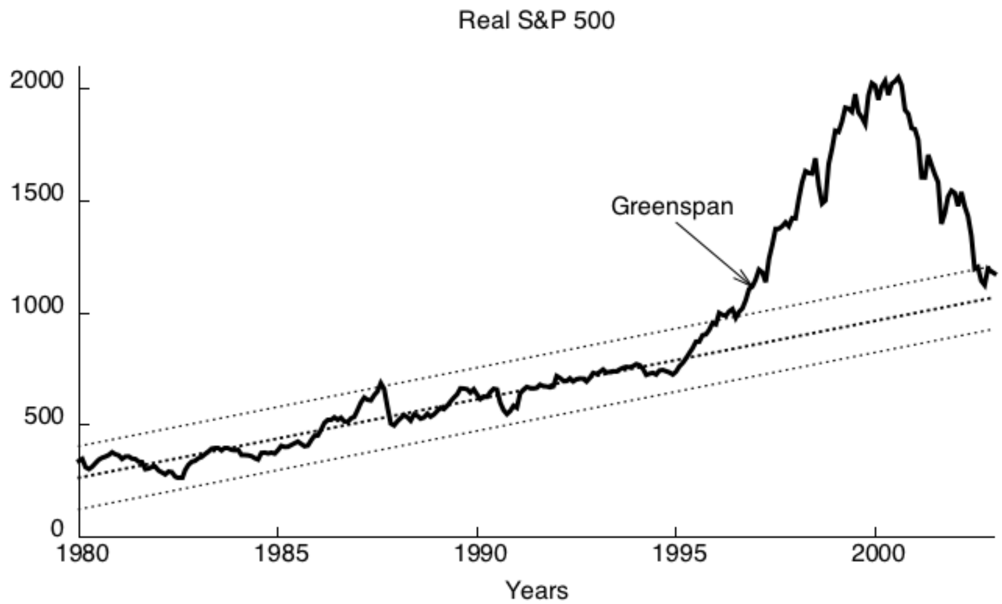

Figure 1: Standard & Poor's 500 stock price (1980–2003) in January 2000 dollars (solid line). Source: Robert J. Shiller's website. Linear trend ±3 S.D. using 1980–1995 data (dotted lines).

extrapolate the most recent trend and enter the market seeking fortune. As outsiders keep entering the market the price grows even higher, at an unsustainable rate. Then, a spiral of speculation develops in which new purchases are no longer driven by the initial good news but rather by the expectation of reselling the assets at an even higher price. As the process continues to feed itself a *mania* could develop, 'a loss of touch with rationality, something close to mass hysteria' in the words of Charles P. Kindleberger, that we shall formalize below as a period in which outsiders are so bullish that a coordinated sell by smart investors could not halt the price run-up. A mania is, thus, the pinnacle of the self-feeding process of speculation; the vicious circle of price increases and subsequent positive feedback trading. As a matter of fact, we borrow the wording from Walter Bagehot's 1856 *Essay on Edward Gibbon*, 'Much has been written about panics and manias, much more than with the most outstretched intellect we are able to follow or conceive; but one thing is certain, that at particular times a great deal of stupid people have a great deal of stupid money.'

The historical record is full of examples of speculative episodes (see Kindleberger and Aliber, 2005) and people often see them as unmissable opportunities to earn exceptional margins on their money. There is a strong incentive to attempt to buy low and sell high, and the strategy that maximizes profits is fairly clear to anyone: enter the market just before it starts to boom and leave it just before it is about to bust. Yet the right timing of moves is not obvious at all and this makes the optimal strategy hard to implement in practice. Even so, many are still aware that boom and bust cycles occur from time to time and guess that those who are attentive enough to buy early on and cautious enough to sell before the eventual collapse actually make lots of money.

Economists have long been intrigued by the recurrence of boom and bust cycles in asset prices. Examples like the 1995–2003 run-up and collapse we see in Figure 1 cannot be foreseeable if they should be consistent with the traditional, rational view on financial markets. That is, these events should take everyone by surprise, follow no particular pattern, and, what is more, prices should at all times track the actual evolution of the asset's real value. Indeed, under full rationality a standard backward induction argument rules out the possibility of predictable boom and bust cycles.

The standard argument is undermined by the evidence against the empirical validity of the backward induction principle (see McKelvey and Palfrey, 1992) as well as by other well-documented behavioral biases (Shiller, 2000a; Shleifer, 2000; Barberis and Thaler, 2003). In spite of this evidence, the profession has insisted on its attempts to build coherent theories of boom and bust cycles robust to the presence of well-financed, fully-rational traders. A big step in this direction is taken in Abreu and Brunnermeier (2003). Their work represents a remarkable advance in our understanding of boom and bust episodes in at least one important dimension: It is the first paper in which smart investors have the collective ability to halt a price run-up induced by irrational traders but nevertheless choose to ride it rather than to lean against it.

The modelling approach of Abreu and Brunnermeier has its roots in the second-generation models of currency attacks. In their model, a critical mass of smart investors is needed to bring prices down—just as a critical mass of speculators is necessary to force a government to abandon a currency peg. The dichotomy between rational and irrational traders in Abreu and Brunnermeier (2003) is akin to those between speculators and the government in the currency attack model of Morris and Shin (1998) and between depositors and commercial banks in the bank run model of Goldstein and Pauzner (2005). In these two later papers it is assumed that there is some region of the fundamentals in which the status quo survives no matter what the actions of the players are. That is, there are states in which the currency peg survives even if all speculators attack it and states in which the bank survives even if all depositors withdraw their funds. In our model below we introduce an identical assumption that allows for states in which the price run-up induced by irrational traders may temporarily survive even if all rational traders sell. Again, a period in which such rather extreme market conditions hold is what we call a mania.

The idea that under short sale constraints stock prices reflect only the most optimistic beliefs goes back at least to Miller (1977). If too many bullish outsiders own stocks, for example, it is then natural to think that market prices will portray their expectations, especially in times when the shorting market works particularly bad. (Recall that to sell short shares one must borrow them first and they are more difficult to borrow if they are held primarily in retail accounts.) Our model studies the implications of the mere possibility of a future mania for the pricing of assets at earlier periods. As in Abreu and Brunnermeier (2003) we assume that rational traders are collectively able to drive prices down at *any state* but nevertheless show that they choose to ride the run-up. In doing so, we dispense with a key assumption in Abreu and Brunnermeier (2003) concerning asymmetric information about the fundamental value of assets that we consider to be problematic. We show existence and uniqueness of a non-degenerate mixed strategy equilibrium under homogeneous information in which all rational traders expect a positive return from speculation and perform comparative statics exersises.

The rest of the paper is organized as follows. In Section 2 we justify our choice of a homogeneous information framework on the idea that a theory that rests solely on asymmetric information may no fit well with what we observe in reality. Section 3 presents a model that is very close to the formulation of Abreu and Brunnermeier (2003) but where there is no asymmetric information and where behavioural traders' bullishness is assumed to be time-varying and random. Section 4 contains our main findings and Section 5 concludes. Proofs and auxiliary results are collected in Appendix A.

2. Homogeneous Information

As pointed out by Eugene F. Fama (Fama, 1970), market efficiency has no empirical implications of its own and we can only test for it within an explicit market model. This means we cannot talk about a bubble on an asset unless we define what its fundamental value is. Economic theorists circumvent this problem by writing down models in which fundamental values are well defined and then study whether bubbles can arise in equilibrium under each particular specification.

In the next section we do precisely that. We present a very stylized, partial equilibrium stock market model and ask whether bubbles may arise in that specific environment. We work within a homogeneous information framework that is particularly hostile to the emergence of bubbles, and we do so because we find that a model in which bubbles arise *only because* there is asymmetric information about the fundamental value of assets presents several empirical challenges. Besides this, we also believe that the main uncertainty that smart investors face is the actual behavior of less sophisticated/irrational investors rather than their peers' opinions about whether a bubble is actually in place. We devote the rest of this section to persuade the reader that our approach is both sensible and worth pursuing.

With rational agents and homogeneous information, asset price bubbles have been ruled out from general equilibrium models under very general conditions (see Santos and Woodford, 1997). Asymmetric information about fundamentals alone neither leads to bubbles under rational expectations (see Tirole, 1982; Milgrom and Stokey, 1982), but combined with short sale constraints it has allowed Allen, Morris, and Postlewaite (1993) and Conlon (2004) to work out examples where bubbles persist because they are not common knowledge among the agents in the economy. These examples, however, are exceptional in that they require fairly specific parameter restrictions to prevent equilibrium prices from revealing the underlying fundamentals.[1]

Abreu and Brunnermeier (2003) proposed a more robust, partial equilibrium model of bubbles based upon a clean and nice story of sequential awareness. In their model, a stock price index departs from fundamentals at a random point in time because of bullish behavioural traders. Rational arbitrageurs become sequentially aware of the mispricing thereafter, but the bubble never becomes common knowledge because no arbitrageur knows where he is in the queue. If the stock price index grows fast and long enough in expectation before the depart from fundamentals, there is a unique equilibrium in which all arbitrageurs choose to ride the bubble.[2]

Robert J. Shiller has collected empirical evidence that does not fit particularly well with the hypothesis of sequential awareness by rational traders. He administered questionnaires to institutional investors between 1989 and 1998 with the aim of quantifying their *bubble expectations*. These are defined in Shiller (2000b) as 'the perception of a temporary uptrend by an investor, which prompts him or her to speculate on the uptrend before the "bubble" bursts.' Shiller's main finding was precisely the absence of the uptrend in the index that would be implied by the sequential awareness hypothesis along the bubble.

Bubbles that last for years also require a large dispersion of opinion among rational traders. Figure 1 displays 1980–2003 real S&P 500 prices and a linear trend with 'confidence

[1] Moreover, because a rational agent holds a bubbly asset only if he expects that asset to be further overpriced in the future, he must believe that other agents will come with a more extreme (optimistic) opinion about its fundamental value as their information gets refined with the passage of time.

[2] Incidentally, this equilibrium requires a (too) high fundamental value. In their model the stock price index $p_t = e^{gt}$ coincides with its fundamental value until a random time t_0 that follows an exponential distribution of parameter λ. If the fundamental value grows at the risk-free interest rate r thereafter, then $g - r > \lambda$ becomes a necessary condition for bubbles (*op. cit.*, p. 180). This implies that the expected present fundamental value of stocks at t_0,

$$\mathbb{E}[e^{-rt_0} p_{t_0}] = \int_0^{+\infty} e^{(g-r)t} \lambda e^{-\lambda t} dt,$$

is infinite because the integral does not converge for $g - r > \lambda$. Our model also dispenses with this infinite-value assumption.

bands' located at three standard deviations. This trend is particularly generous because it was computed using data from the bull market of the eighties and early nineties. If, being further conservative, we placed the birth of the bubble at the point labelled 'Greenspan' in Figure 1,[3] the time elapsed before the 2000 peak would amount to almost four years. The question of whether there was enough dispersion of opinion to sustain a four-year long bubble may have no satisfactory answer, however, as we are dealing here with highly stylized models from where it would be unfair to draw quantitative predictions.

But our case for homogeneous information goes beyond the sequential awareness hypothesis and concerns the role asymmetric information about fundamentals in general as the primary source of bubbles. First, if bubbles persist solely on the basis that they are not common knowledge among rational traders, then a public disclosure of the fact that assets are overpriced should cause their immediate burst. Kindleberger and Aliber (2005, ch. 5) document that the historical record provides little evidence supporting this claim. Virtually every bubble has been accompanied by unsuccessful public warnings by either government officials or members of the business establishment. A famous example of this was Alan Greenspan's statement on 5 December 1996 that the US stock market was 'irrationally exuberant.' Indeed, a glimpse at Figure 1 could make us think that the Fed chairman's warning was—if anything—more an encourager than an deterrent.

Second, there are instances in which the cash flows of various stocks are linked via an explicit and publicly known formula that actual prices do not comply with. In these cases any deviation from the theoretical par is by definition a bubble that cannot be attributed to asymmetric information about their relative fundamental values. Such opportunities for unambiguous relative price comparisons are rare, but they constitute an exceptional tool for discerning among the various potential sources of mispricing.

Lamont and Thaler (2003) study a sample of carve-outs in which the parent firm has stated its intention to spin off its remaining shares of a subsidiary firm. They focus particularly in the case of Palm and 3Com. 3Com sold a 5% of Palm on March 2, 2000 and announced a spin off by which 3Com shareholders would receive 1.5 shares of Palm for every share of 3Com that they owned. This implied that the price of 3Com shares should be at least 1.5 times the price of Palm shares. However, the price of Palm shares experienced—for months—a bubble that involved a negative value for 3Com's non-Palm assets and business.[4]

[3] The 'Greenspan point' is further explained below.

[4] Another piece of evidence provided by Lamont and Thaler is that holders of Palm shares were paying much more than they could pay using the options market, probably meaning that those investors were ignorant of the options market and unaware of the cheaper alternative. Their narrative also seems to fit well with the concept of mania outlined in the introduction: 'In the case of Palm, arbitrageurs faced little risk but could not find enough shares of Palm to satiate the demands of irrational investors.'

Another source of examples is provided by the so called 'siamese-twin' securities. The most famous case is probably that of Royal Dutch and Shell, but there are others (cf. Froot and Dabora, 1999). These two firms merged their interests in 1907 on a 60:40 basis while remaining separate and distinct entities. It is information of public domain that the cash flows of both firms are split in these proportions, which implies that the price of Royal Dutch shares should be equal to 1.5 times the price of Shell shares. Still, as in the example of Palm and 3Com above, huge relative price deviations from the theoretical par have been the rule rather than the exception in the markets they are traded.

Lamont and Thaler point at the case of Palm and 3Com to underscore the role of short sale constraints, whereas Shleifer (2000, ch. 2) interprets the case of Royal Dutch and Shell as evidence supporting the importance noise trader risk. Our model will combine both features, but for now our main message is that neither bubble can be attributed to a rational disagreement about fundamentals because in these examples cash flows are explicitly tied together.

Third, there is strong evidence showing that lack of common knowledge of fundamentals is unnecessary to cause asset price bubbles in experimental settings. Smith, Suchanek, and Williams (1988) show that bubbles can emerge in a setting where the probability distribution of dividends is commonly known to all participants. Their results are robust; they have been replicated many times and under various experimental designs. Lei, Noussair, and Plott (2001) went a step further and studied whether it was lack of common knowledge of rationality itself—rather than of fundamentals—what was driving earlier experimental findings. They found that only actual irrational behavior could explain many of the bubbles they observed.

Our approach in this paper may be illustrated by a famous quote attributed to Sir Isaac Newton (Malkiel, 1985), who surely qualifies as one of the most rational men of his time. After being caught by the burst of the South Sea Bubble, in which share prices of the South Sea Company went up and down almost tenfold in less that a year, Newton himself exclaimed, 'I can calculate the motions of heavenly bodies, but not the madness of people.' In our view, even more important than what Newton says in this widely-cited quote is what he does not say; his concerns about the root of his financial failure were clearly not on any uncertainties surrounding the actual worth of the Company, neither on how this information was disseminated across investors. Much on the contrary, he put the blame of his misfortune on his own inability to predict the irrational behavior of the multitude.

3. The Model

Our framework borrows from the continuous-time model of Abreu and Brunnermeier (2003). We consider a market for stocks. There is a continuum of mass $0 < \mu < 1$ of arbitrageurs who seek to maximize the expected discounted value of their transactions. They can sell and buy back shares at any time at the discounted cost $0 < c < 1$ per transaction, but they are constrained on the maximum long and short positions they can take. In particular, the selling pressure exerted by each arbitrageur must lie within the unit interval at any time. There is also a continuum of mass 1 of behavioral traders whose trading behavior is given exogenously and summarized by their aggregate absorption capacity κ.[5] Arbitrageurs cannot observe the absorption capacity of behavioral traders.

The fundamental value of stocks is e^{rt} for all $t \geq 0$. The *pre-crash* price is e^{gt}, with $g > r > 0$. The market price is equal to the pre-crash price as long as the aggregate selling pressure s of arbitrageurs stays below the absorption capacity κ of behavioral traders. As soon as this ceases to happen, the market price drops to its *post-crash* level e^{rt}.

We interpret the market price process as follows: A series of unanticipated good news show before $t = 0$ that justify the higher growth rate g. Thereafter, the higher rate is no longer justified by fundamentals, which now grow at rate r. A *bubble* starts at $t = 0$ that persists until the selling pressure by arbitrageurs equals (or exceeds) the absorption capacity of behavioral traders.

The absorption capacity κ is a function of time and of a random variable X with standard uniform distribution. Random variable X is a hidden state variable, but its probability law is common knowledge among arbitrageurs. For each positive realization x of X, the absorption capacity is assumed to be given at any time by the formula:

$$\kappa(x,t) := \begin{cases} x \sin\left(\frac{t}{x}\right) & \text{if } 0 \leq t < \pi x \\ 0 & \text{if } t \geq \pi x, \end{cases} \tag{1}$$

where $\pi = 3.1415\ldots$ For continuity, we also assume that $\kappa(0,t) := 0$ for all $t \geq 0$. Figure 2 displays several sample paths of $\kappa : [0,1] \times \mathbb{R}_+ \longrightarrow [0,1]$ (for states $x = 0.2, 0.4, 0.6, 0.8, 1.$). It is a continuous function, nondecrasing in x for t fixed and cuasiconcave in t for fixed x. The shape of the sample paths of κ reflects the idea that, as long as the bubble persists, more and more money from behavioral traders enters the market until the maximum established by state variable X is reached; at this point, the process reverses. This aggregate behavior may be thought as resulting from individual entry-exit strategies of behavioral traders that

[5] The assumption that there is a mass of size 1 of behavioural traders and a mass $\mu \leq 1$ of rational traders is also in DeLong, Shleifer, Summers, and Waldmann (1990).

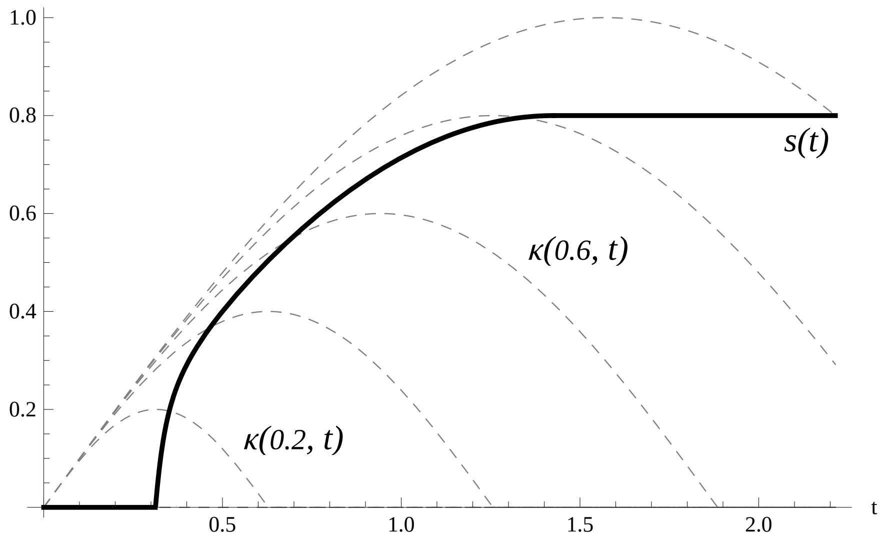

Figure 2: Various trajectories of $\kappa(\cdot, t)$ (dashed lines) and the equilibrium aggregate selling pressure $s(t)$ for parameter values $\mu = 0.8$ and $g - r = 0.1$ (solid line).

realize that the bubble opens a temporary window for speculation but do no enter into more sophisticated backward-induction chains of reasoning. Larger states correspond to more aggressive behavior from the part of behavioral traders; for larger values of x two things happen: (i) more money flows into the market, and (ii) it stays in for longer. These allow to interpret state variable X as an index of the behavioral traders' bullishness along the bubble.

Arbitrageurs get no information before the crash. On the other hand, the best they can do after the crash is to quit and never re-enter the market—because all transactions made after the crash are costly and worthless. There are no contingencies to which they may have to adapt besides the crash itself, and the adaptation to this event is trivial. This means that an arbitrageur's strategy only has to specify a set of orders that will be placed sequentially along the bubble. Because transactions are costly, each arbitrageur plans, at most, a finite amount of them.

A pure strategy profile is a function $\sigma : [0, \mu] \times \mathbb{R}_+ \longrightarrow [0, 1]$ that specifies the selling pressure $\sigma(i, t)$ exerted by each arbitrageur $i \in [0, \mu]$ at each instant $t \in \mathbb{R}_+$. Without any loss of generality, we assume that all arbitrageurs start at their maximum long position, i.e., $\sigma(i, 0) = 0$ for all i. Their aggregate selling pressure is

$$s(t) := \int_0^\mu \sigma(i, t)\, di.$$

A *trigger-strategy* for an arbitrageur specifies a unique transaction at which he completely sells out. The set of trigger-strategies is thus indexed by the time at which each such sale occurs. For example, if each arbitrageur i plays some trigger-strategy t_i, the profile $\sigma(i,t) = \mathbf{1}_{[t_i,+\infty)}(t)$ obtains. A mixed trigger-strategy is a mixed strategy which only contains trigger-strategies in its support. If each arbitrageur independently draws a trigger-strategy from the same distribution function F, the corresponding aggregate selling pressure is $s(t) = \mu F(t)$ almost surely for all $t \geq 0$.

We define the *date of burst* of the bubble given the absorption capacity κ and the aggregate selling pressure s.

Definition 1. The date of burst of the bubble is the random variable

$$T(x) := \inf\{t : s(t) \geq \kappa(x,t), t > 0\}. \tag{2}$$

The market price jumps from the pre-crash to the post-crash level at the date of burst, i.e.:

$$p(x,t) := \begin{cases} e^{gt} & \text{if } t < T(x) \\ e^{rt} & \text{if } t \geq T(x). \end{cases}$$

All transactions take place at the market price p. This assumption may seem questionable if there are states in which the limit from the left of the aggregate selling pressure is strictly smaller than the absorption capacity at the date of burst. It would be more natural to assume that some of the orders placed at the date of burst, up to the limit imposed by the outstanding absorption capacity at that moment, are executed at the pre-crash price. Our assumption simplifies the analysis and does not affect the results.

For each state $x > \mu$ there is a non-empty interval of time wherein the absorption capacity exceeds the maximum aggregate selling pressure. Formally, $I_\mu(x) := \{t : \kappa(x,t) > \mu\} \neq \emptyset$ for $\mu < x \leq 1$. We call such interval a mania. A mania is, thus, a period in which not even a coordinated attack from the part of arbitrageurs can burst the bubble. We shall see below that if a mania could not happen, the bubble would always collapse at $t = 0$. Its significance comes form the fact that, whereas the abnormal price growth $g > r$ gives symmetric incentives to stay in the market to all arbitrageurs, the possibility of occurrence of a mania allows them to choose asymmetric exit strategies in equilibrium.

4. Symmetric Equilibria in Trigger-Strategies

Arbitrageurs are fully rational players. They form a rational conjecture about the date of burst and choose their preferred trading strategy accordingly. The second step is straightforward, but to form a rational conjecture about the date of burst requires a good deal of strategic thinking.

We show that there exist symmetric equilibria in mixed trigger-strategies. These equilibria are characterized by a mixed trigger-strategy F such that—if everyone else plays it—each arbitrageur finds every strategy in its support optimal. The mixed trigger-strategy F determines the aggregate selling pressure which, trough (2), defines the date of burst. Since there is a continuum of arbitrageurs, none can affect the date of burst, which is the only way in which the payoff of an arbitrageur may be affected by the choices of others. Hence, the problem of an arbitrageur is to choose a best response given the distribution of the date of burst induced by F. A symmetric equilibrium is found if every strategy in the support of F is indeed a best response.

The payoff from the trigger-strategy t is

$$\begin{aligned} v(t) :&= E\left[e^{-rt}p(X,t) - c\right] \\ &= e^{(g-r)t}[1 - G(t)] + G(t) - c, \end{aligned} \tag{3}$$

where G denotes the distribution function of the date of burst.[6] Equation 3 neatly expresses the arbitrageurs' trade-off between staying in and exiting the market. On one side, the discounted pre-crash price grows with time; on the other side, the probability of survival of the bubble decreases. Optimal trigger-strategies fulfill the first-order condition $v'(t) = 0$. We may also write this as

$$h(t) = \frac{g-r}{1-e^{-(g-r)t}},$$

where $h(t)$ denotes the hazard rate that the bubble will burst at t.[7] This equation will hold for all t within the support of the non-degenerate equilibrium mixed trigger-strategy of Proposition 2 below.

4.A. Pure strategies

Our first result is rather unsurprising. If all other arbitrageurs play the trigger-strategy $t = 0$, the bubble bursts immediately. Any transaction takes place at the post-crash price and hence yields at most $1 - c$. This shows that the trigger-strategy $t = 0$ is indeed a best response, which characterizes a symmetric equilibrium.

Proposition 1. *There is a unique symmetric equilibrium in pure trigger-strategies. In this equilibrium each arbitrageur sells out at $t = 0$.*

It is easy to see that there is no other symmetric equilibrium in pure trigger-strategies

[6] An expression for the payoff from a general trading strategy is given in the proof of Proposition 2 in Appendix A.

[7] This is also the marginal sell-out condition in Abreu and Brunnermeier (2003).

by the usual backwards induction argument. No such equilibrium exists for $t \geq \pi$ because the bubble never bursts after π. When everyone else sells at the same date $0 < t < \pi$ there is an upward jump in the probability distribution of the date of burst at t. An arbitrageur who deviates and sells a bit earlier sacrifices an infinitesimal reduction in the pre-crash price in return for a discrete decrease in the probability of burst. This shows that selling at the same time as others do cannot be a best response.

There are uncountably many equilibria in which the bubble bursts at $t = 0$. For instance, there are uncountably many equilibria in which every arbitrageur plays a mixed trigger-strategy that puts strictly positive mass on $t = 0$. From now on, we restrict the analysis to equilibria in which the bubble has some chance of survival, that is, to equilibria in which $G(0) < 1$.

The model thus keeps the standard efficient-market hypothesis (EMH) solution: despite the potential speculative profits, there are equilibria in which competition among arbitrageurs causes an early burst and no one benefits from the bubble. The bubble is extremely weak in its infancy—when virtually any selling pressure can make it burst—and its survival requires coordination among arbitrageurs. They have the opportunity to feed the bubble, perhaps all the way to a mania, but the fear that others may not concur can ruin it all.

4.B. Non-degenerate mixed strategies

Arbitrageurs do not have to content themselves with this solution. It is possible to reconcile the individual incentive to time the market with the collective interest in feeding the bubble. The ingredient that makes it possible is the possibility of occurrence of a mania. If a mania could not happen, an arbitrageur would stay in the market only if he believed that a sufficiently big mass of arbitrageurs would stay in as well. However, these beliefs cannot be held uniformly across the population because they are incongruent. But if a mania can take place some arbitrageurs may find it optimal to stay invested as long as they believe that the bubble will not be killed too soon. The reason is that arbitrageurs are no longer competitors during a mania.

How can these arbitrageurs rationally believe that the rest will not kill the bubble? We show below how this coordination problem is resolved in equilibrium: Arbitrageurs leave the market at a rate that allows the outbreak of manias.

The shape of function G in (3) determines what is optimal for an arbitrageur. We know that for a mixed trigger-strategy F to characterize an equilibrium, all pure strategies in the interior of its support must be optimal. This imposes restrictions on G that translate into restrictions on F trough function T. The following result states two properties of the date of burst T that we use to show how individual rationality restricts the shape of any equilibrium mixed trigger-strategy F.

Lemma 1. *If there is a symmetric equilibrium in mixed trigger-strategies that fulfills $G(0) < 1$, then function T is strictly increasing and continuous.*

The previous lemma tells us how to obtain the distribution of the date of burst from the distribution of the state variable:

$$G(t) := \mathbf{P}(T(X) \leq t) = \mathbf{P}(X \leq T^{-1}(t)) = T^{-1}(t). \tag{4}$$

Corollary 1. *Given the conditions of Lemma 1, $G(t) = T^{-1}(t)$ for all $t \leq \pi - \arcsin(\mu)$.*[8]

Restrictions on G thus translate into restrictions on T^{-1}. Let us label the inf and the sup of the support of F as $\underline{t}$ and $\bar{t}$ (Lemma 5 in Appendix A shows that the support of F is indeed an interval). From Corollary 1, we rewrite function v as

$$v(t) = e^{(g-r)t}\left[1 - T^{-1}(t)\right] + T^{-1}(t) - c.$$

Let us label the equilibrium payoff as v^*. Since v^* is the payoff to each strategy inside the equilibrium support we must have

$$T^{-1}(t) = \frac{e^{(g-r)t} - (v^* + c)}{e^{(g-r)t} - 1} \tag{5}$$

for all $\underline{t} \leq t \leq \bar{t}$ (Lemma 4 in Appendix A shows that this equation holds also at the endpoints). Note that, because $v^* + c > 1$, the right-hand side of (5) defines a differentiable, strictly increasing function of time.

Lemma 2. *Given the endpoints $\underline{t}$ and $\bar{t}$ of the equilibrium support, function T^{-1} is uniquely determined for all $t \leq \pi - \arcsin(\mu)$.*

Proof. We already have a closed-form expression for function $T^{-1}(t)$ in $[\underline{t}, \bar{t}]$ that is given by (5). Because no arbitrageur sells before $\underline{t}$, the bubble bursts when the absorption capacity returns to zero for all states $x < \underline{t}/\pi$ (see Figure 2). This means that function T^{-1} is equal to t/π for all $t < \underline{t}$. Because T is strictly increasing and continuous by Lemma 1, some price path must decrease at each $\bar{t} < t \leq \pi - \arcsin(\mu)$. This implies that the sample path of κ that touches s at $\bar{t}$ cannot be strictly increasing at $\bar{t}$. Otherwise, because s is flat for $t \geq \bar{t}$, there would be some interval $(\bar{t}, \bar{t} + \epsilon]$ (with $\epsilon > 0$) in which no price path decreases,

[8] We restrict arcsin to its principal branch hereafter. Quantity $\pi - \arcsin(\mu)$ corresponds to the second time at which function $\kappa(1, t)$ is equal to μ. Because κ is smaller than μ for all x and $t > \pi - \arcsin(\mu)$ no bubble bursts beyond this point.

contradicting Lemma 1. Given the shape of κ, each higher path must intersect s for the first time while it is decreasing. Hence, $T^{-1}(t)$ is the solution x to $\kappa(x,t)=\mu$ for all t in $(\bar{t}, \pi - \arcsin(\mu)]$. This completes the proof. □

The next lemma shows that the endpoints $\underline{t}$ and $\bar{t}$ of the equilibrium support are both unique.

Lemma 3. *All F fulfilling the conditions of Lemma 1 have the same support.*

Lemma 2 and Lemma 3 imply that function T is unique for the class of equilibria we consider. Hence, to find an equilibrium within this class amounts to find a mixed trigger-strategy F which, trough (2), induces such T. Our main result is that there is a unique mixed trigger-strategy that does the job.

Proposition 2. *There is a unique equilibrium fulfilling the conditions of Lemma 1. In this equilibrium, each arbitrageur plays the mixed trigger-strategy*

$$F(t) = \frac{1}{\mu}\kappa\left(T^{-1}(t), t\right) \tag{6}$$

for all $t \leq \pi - \arcsin(\mu)$.

The equilibrium function s displayed in Figure 2 shows how everything fits together. The bubble bursts at the point at which the realized sample path of κ crosses s. Some paths cross s while they are increasing, whereas others cross it while they are decreasing. The key to understand the persistence of bubbles is to notice that some paths corresponding to states $x > \mu$ are of the second kind, that is, that manias can indeed occur in equilibrium. Arbitrageurs start selling at $\underline{t}$, but their aggregate selling pressure accumulates slowly enough to feed the bubble, to let it grow.

Some arbitrageurs sell earlier than others. Those who rush out of the market have more chances to sell at the pre-crash price, and those who wait have the opportunity to earn higher profits; but all expect the same payoff ex-ante. The coordination achieved in equilibrium is remarkable, but imperfect. Most sample paths of κ cross s before they reach their maximum, which means that arbitrageurs as a whole could have done it better because there are still some behavioral traders willing to inject money into the market. This is the curse of competition, the same that may induce an immediate collapse as in Proposition 1.

We should add a remark on the way we interpret the equilibrium in Proposition 2. We have focused on symmetric equilibria though it is easy to see that there are uncountably many asymmetric equilibria that share the same s. The reason is that we find that an asymmetric equilibrium is an unnatural solution concept within our essentially symmetric context. We would find it hard to justify why otherwise identical arbitrageurs played different strategies in equilibrium, how did they know which strategy should they play, and so on. But symmetry cannot be more than a rough approximation to reality, however appealing and convenient. Asymmetries surely play their role in the workings of the stock market, though we do not bring them to the core of the discussion. We interpret our mixed strategy equilibrium from the Bayesian perspective, that is, from the view that it serves as an approximation to a more complex world in which each arbitrageur harbors doubts about privately known characteristics of other arbitrageurs. Rather than to a classical randomizing interpretation of mixed strategies, we subscribe to the modern view in which arbitrageurs would be in fact playing pure strategies, with mixed-strategies representing their uncertainties about others. Because every arbitrageur is negligible, there is no reason for anyone to conceal his action.

4.C. Comparative statics

There are three parameters of interest in our model: c, $g-r$, and μ. It can be readily seen from (3) that parameter c does not affect marginal utility and, therefore, does not affect choice. Our numerical exercises corroborate the commonsensical intuition that an increase in $g-r$ encourages arbitrageurs to ride the bubble. In particular, we see that the equilibrium strategy (6) shifts to the right as $g-r$ goes up—although the effect is milder for large values of parameter μ.

The most interesting quantity is $1-\mu$, which may be interpreted as the probability of a mania taking place. An increase in μ discourages riding the bubble. Further, it is straightforward to prove that the equilibrium F and s converge to $\kappa(1,t)$ as μ approaches one—rather than to $\mathbf{1}_{[0,+\infty)}(t)$. That is, the equilibrium in Proposition 2 is robust in the sense that it does not converge to the standard pure-strategy equilibrium of Proposition 1 as the probability of a mania goes to zero.

Proposition 3. *In the limit, as μ approaches 1, the equilibrium mixed trigger-strategy F and the corresponding aggregate selling pressure s converge to the function* $\sin(t)$ *for all* $t \leq \pi/2$.

5. Concluding Remarks

In this paper we present a game-theoretic model of stock price bubbles. Our model is intended to formalize a widely-held view on how the strategic side of a bubble is perceived by sophisticated investors. Arbitrageurs know the fundamental value of stocks but are uncertain about behavioral trading. This prevents them from foreseeing the crash, yet all ride the bubble for some time. The belief on a rising bubble is self-fulfilling and allows some arbitrageurs to profit from behavioral traders—and from other arbitrageurs as well. Some arbitrageurs sell earlier than others, and the gains from waiting are just compensated by the increase in the probability of a crash. In short, a bubble is (roughly) a positive-sum, risky game in which sophisticated investors extract rents from less sophisticated investors.

Proposition 1 and Proposition 2 provide two legitimate solutions of the game. Proposition 2 hints on the logic of the persistence of bubbles and suggests how this phenomenon can be reconciled with a good deal of rationality in the market. The standard EMH equilibrium in Proposition 1 loses strength in comparison since the chances are that arbitrageurs will wait to see whether a bubble rises up. Further, the standard equilibrium washes out if we allow the market price to undershoot the fundamental value at the date of burst.

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A. Proofs

We use two properties of functions G and v repeatedly. First, because G is a distribution function, it is right-continuous, which, in turn, implies that v is also right-continuous. Second, v is increasing whenever G is constant and vice versa.

A.A. Preliminary results

Lemma 4. *Given the conditions of Lemma 1, $v(\underline{t}) = v(\bar{t}) = v^*$.*

Proof. If either the infimum or the supremum is an isolated point of the support, then it is a mass point of F, and v at that point must be v^*.

Because v is right-continuous, $\lim_{t\downarrow\underline{t}} v(t) = v(\underline{t})$, which, if $\underline{t}$ is not isolated, implies $v(\underline{t}) = v^*$.

Suppose that $v(\bar{t}) < v^*$. If $\bar{t}$ is not isolated, this can happen only if v has a downward jump at $\bar{t}$, which, by (3), only occurs if G jumps at $\bar{t}$. G jumps only if s (and F) also have a jump at the same point; a point in which s first surpasses a strictly positive mass of sample paths of κ. But this means that $\bar{t}$ is a mass point of F, and so $v(\bar{t}) = v^*$. □

Lemma 5. *Given the conditions of Lemma 1, $\underline{t} > 0$, and the support of F is an interval.*

Proof. We know from Proposition 1 that F is non-degenerate. Because G is right-continuous and $G(0) < 1$, we know that for every $\epsilon > 0$ there must exist some δ such that $t < \delta$ implies that $G(t) < G(0) + \epsilon$. Take any $\epsilon < 1 - G(0)$ and see that $v(\delta) > 1 - c$. This shows that $v(\underline{t}) = v^* > 1 - c$, and so $\underline{t} > 0$.

We use the following properties of function s. We must have both $s(t) < \kappa(1, t)$ for all $0 < t \leq \arcsin(\mu)$ and $s(t) = \mu$ for all $t \geq \pi - \arcsin(\mu)$. Otherwise, the bubble bursts no later than $\bar{t}$ with probability one, which would imply $v(\bar{t}) = v^* = 1 - c$.

We show now that F is strictly increasing for all $\underline{t} \leq t \leq \bar{t}$. Suppose that $F(t_0) = F(t_1) = k_0/\mu$ for some $\underline{t} \leq t_0 < t_1 \leq \bar{t}$, that is, $s(t) = k_0$ for all $t \in [t_0, t_1]$. We will see that this leads to a contradiction. Let $t_- = \inf\{t|F(t) = k_0/\mu\}$, that is, t_- belongs to the support of F. Let x_- be the solution to the equation $\kappa(x, t_-) = k_0$ (that this equation has a solution is implied by the previous paragraph). The derivative of $\kappa(x_-, t)$ with respect to t, evaluated at t_-, cannot be strictly positive. If this were the case, there would exist some $\epsilon > 0$ such that $s(t) = k_0 < \kappa(x_-, t)$ for all $t_- < t \leq t_- + \epsilon$, which, in turn, would imply that no market price path decreases therein. But this contradicts that fact that t_- belongs to the equilibrium support; any point fulfilling $t_- < t \leq t_- + \epsilon$ pays more than $v(t_-) = v^*$ because $G(t) = G(t_-)$.

Let $t_+ = \sup\{t|F(t) = k_0/\mu\}$, that is, t_+ belongs to the support of F. Since some price path has to decrease in $(t_-, t_-+\epsilon]$ for every $\epsilon > 0$, and since the aforementioned derivative has to be either negative or zero, we have that a price path decreases at each instant $t_- < t < t_+$, that is, a sample path of κ intersects s for the first time at each of these instants. Let x_+ be solution to the equation $\kappa(x, t_+) = k_0$. We have just shown that function T is

$$T(x) = \left[\pi - \arcsin\left(\frac{k_0}{x}\right)\right] x$$

for all $x_- < x < x_+$. Therefore, we can make the change of variable $t = T(x)$ and rewrite the payoff function v for $t_- < t < t_+$ as

$$e^{(g-r)T(x)}(1-x) + x - c, \tag{7}$$

where $x_- < x < x_+$. Since function (7) is uni-modal for $k_0 \leq x \leq 1$ (provided that $k_0 > 0$), we must have that v is decreasing for $t_- < t < t_+$; otherwise, the payoff would not be maximized at t_-. The payoff function v can only have downward jumps (which correspond to upward jumps of G) and, therefore, we must have that $v(t_-) > v(t_+)$, which contradicts the fact that t_+ belongs to the equilibrium support. This shows that F is strictly increasing in $[\underline{t}, \bar{t}]$. □

Lemma 6. *Given the conditions of Lemma 1, G is continuous and strictly increasing for all $t \leq \pi - \arcsin(\mu)$.*

Proof. Since $s(t) = 0$ for all $t < \underline{t}$, we have that $T(x) = \pi x$ for all $x < \underline{t}/\pi$, which implies that $G(t) = t/\pi$ for all $t < \underline{t}$. Thus, G is strictly increasing and continuous for all $t < \underline{t}$.

Suppose that there were $\underline{t} \leq t_0 < t_1 \leq \bar{t}$ such that $G(t_0) = G(t_1) = h_0$. Then, $v(t_0) < v(t_1)$, contradicting the fact that $v(t_0) = v^*$, as implied by the previous lemma. Also, if G had a jump at some $\underline{t} \leq t_0 \leq \bar{t}$, then v would have a downward jump at t_0, which cannot happen if $v(t_0) = v^*$. Thus, G is strictly increasing and continuous for all $\underline{t} \leq t \leq \bar{t}$.

There cannot be any $\epsilon > 0$ such that no price path decreases in $(\bar{t}, \bar{t} + \epsilon]$ because that would mean that $G(\bar{t} + \epsilon) = G(\bar{t})$, that is, $v(\bar{t} + \epsilon) > v(\bar{t})$, which contradicts the fact that $v(\bar{t}) = v^*$. Since s is flat for all $t \geq \bar{t}$, any sample path of κ which intersects s for the first time whithin that range has to be decreasing. This implies that one price path decreases at each $\bar{t} < t \leq \pi - \arcsin(\mu)$. Thus, G is strictly increasing and continuous for all $\bar{t} < t \leq \pi - \arcsin(\mu)$. □

Proof of Lemma 1. T is monotone increasing because $\kappa(x_1, t) \geq \kappa(x_0, t)$ for all t and all $x_1 > x_0$. Let us define

$$\xi(t) := \sup \{x | T(x) \leq t\}.$$

It is clear from the definition that $T(x) \leq t$ implies $x \leq \xi(t)$ and, hence, $\mathbf{P}(T(X) \leq t) \leq \mathbf{P}(X \leq \xi(t))$. If $x < \xi(t)$, then $T(x) \leq t$; since T is monotone increasing, $T(x) > t$ would contradict the definition of $\xi(t)$. Therefore, there is at most one point which could satisfy both $x \leq \xi(t)$ and $T(x) > t$, namely $x = \xi(t)$. Because the standard uniform is a continuous distribution, any singleton has probability zero. Thus, we can write:

$$G(t) = \mathbf{P}(T(X) \leq t) = \mathbf{P}(X \leq \xi(t)) = \xi(t). \tag{8}$$

Since T is monotone increasing, it can only have jump discontinuities. Suppose that T had a discontinuity at x_0 and let $t_0 < t_1$ be the one-sided limits of T as x approaches to x_0. Then, we should have that $\xi(t) = \xi(t_0)$ for all $t_0 \leq t < t_1$, which, by Lemma 6 and (8), cannot happen (ξ is strictly increasing for all $t \leq \pi - \arcsin(\mu)$). This shows that T is continuous.

To show that T is strictly increasing, suppose that $T(x_0) = T(x_1) = t_0$ for some $x_0 < x_1$. Let $x_- = \inf\{x | T(x) = t_0\}$ and $x_+ = \sup\{x | T(x) = t_0\}$. Because T is increasing, $\xi(t_0) = x_+$ and $\xi(t_0 - \epsilon) \leq x_-$ for all $\epsilon > 0$. This means that ξ must have a discontinuity at t_0, which cannot happen since Lemma 6 together with (8) imply that ξ is continuous.

Since T is strictly increasing and continuous, $\xi = T^{-1}$.

Proof of Lemma 3. Using the change of variable $t = T(x)$, we can write

$$v(T(x)) = e^{(g-r)[\pi - \arcsin(\mu/x)]x}(1 - x) + x - c \tag{9}$$

for all $x \geq T^{-1}(\bar{t})$. This function is uni-modal for $\mu \leq x \leq 1$. Since $v(t)$ must be smaller or equal than v^* for all $t > \bar{t}$, we know that $T^{-1}(t)$ cannot be smaller than the point $\overline{x}$ maximizing (9). Otherwise, (9) would be strictly increasing at $T^{-1}(\bar{t})$, which contradicts $v(\bar{t}) = v^*$.

On the other hand, we know that $s(t) \leq \mu$ for all t. This implies that, for all $x > \mu$, the inequalities $\arcsin(\mu/x)x < T(x) < [\pi - \arcsin(\mu/x)]x$ cannot both be true: within the interval of time $(\arcsin(\mu/x)x, [\pi - \arcsin(\mu/x)]x)$—the mania—the absorption capacity is strictly greater than μ for all $x > \mu$, which means that no price path can decrease therein. The inequality $T(x) \leq \arcsin(\mu/x)x$ can neither be true. Suppose that there exists some $x_0 >$

μ such that $T(x_0) \leq \arcsin(\mu/x_0)x_0$. We know that for all $x > x_0$ and all $\arcsin(\mu/x_0)x_0 \leq t \leq [\pi - \arcsin(\mu/x_0)]x_0$, $\kappa(x,t) > \mu$. Therefore, no price path decreases within the interval of time $(\arcsin(\mu/x_0)x_0, [\pi - \arcsin(\mu/x_0)]x_0)$, which is incompatible with T being continuous. In short, we must have that $T(x) \geq [\pi - \arcsin(\mu/x)]x$ for all $\mu < x \leq T^{-1}(\bar{t})$. Going back to v, we must have

$$v(T(x)) \geq e^{(g-r)[\pi - \arcsin(\mu/x)]x}(1-x) + x - c \tag{10}$$

for all $\mu < x \leq T^{-1}(\bar{t})$. This implies that $T^{-1}(\bar{t})$ cannot be greater than $\bar{x}$, since $\bar{x}$ would then contradict the inequality. Hence, there is only one admissible $T^{-1}(\bar{t})$, namely $\bar{x}$, and $\bar{t} = [\pi - \arcsin(\mu/\bar{x})]\bar{x}$.

Clearly, $\bar{t}$ gives v^*. On the other hand, $\underline{t}$ is the solution to

$$e^{(g-t)t}\left(1 - \frac{t}{\pi}\right) + \frac{t}{\pi} - c = v^* \tag{11}$$

for $t < \bar{t}$. Such solution always exists and is unique because the left-hand side of (11), with the change of variable $t = \pi x$, is uni-modal in $[0,1]$, equal to zero at $t = 0$, and lies above the right-hand side of (9) for all $x > \mu$.

Proof of Proposition 2. The proof consists of two parts. In the first part we show that function F characterizes an equilibrium in mixed trigger-strategies. In the second part we show that this is the only equilibrium.

Part I: For the first part, we first show that F is indeed a distribution function and then show that F induces the equilibrium date of burst T.

Function F is a distribution function: Let endpoints $\underline{t}$ and $\bar{t}$ be as defined in the proof of Lemma 3, and let function T^{-1} be as defined in the proof of Lemma 2. Suppose that the aggregate selling pressure is

$$s(t) = \kappa(T^{-1}(t), t),$$

for $t \leq \pi - \arcsin(\mu)$. By Definition 1, the date of burst in this case is T and its distribution function is $G = T^{-1}$ by (4) (which, recall, only requires T to be strictly increasing and continuous, not the existence of a trigger-strategy equilibrium). We saw in the proof of Lemma 3 that $v(t)$ is then strictly increasing for all $t < \underline{t}$ and strictly decreasing for all $t > \bar{t}$. That is, all trigger-strategies outside the equilibrium support pay less than the equilibrium payoff v^*. It only remains to show that no other pure strategy pays more that v^*. Consider an arbitrary pure strategy involving $N > 1$ transactions (since all arbitrageurs liquidate a some time, every pure strategy which is not a trigger-strategy involves more than one transaction). Let $(\mathbf{t}, \mathbf{z})$ be the vector specifying the transaction dates $(t_1, \ldots, t_N)$ and the positions held between transactions $(z_1, \ldots, z_N)$, where $z_N = 1$. Such strategy is a plan of action which says what the arbitrageur will do along the bubble. If the bubble bursts between two transaction dates, the arbitrageur liquidates. The payoff of a general pure strategy involving finitely many transactions is, therefore,

$$V((\mathbf{t}, \mathbf{z})) := \sum_{n=1}^{N} \left[(z_n - z_{n-1})e^{(g-r)t_n} - c\right][1 - G(t_n)]$$
$$+ (1 - z_{n-1} - c)[G(t_n) - G(t_{n-1})]\mathbf{1}_{[0,1)}(z_{n-1}),$$

where $z_0 = t_0 = 0$. It is a matter of algebra to show that

$$V((\mathbf{t}, \mathbf{z})) < \sum_{n=1}^{N}[z_n - z_{n-1}]v(t_n)$$

whenever $(\mathbf{t}, \mathbf{z})$ does not correspond to a trigger-strategy. We can rewrite the right-hand side as

$$v(t_N) + \sum_{n=1}^{N-1} z_n[v(t_n) - v(t_{n+1})].$$

Now let $N_1 = \max\{n|t_n \leq \bar{t}\}$. Since v is nondecreasing for all $t < \bar{t}$, we have that the last expression is bounded above by

$$v(t_N) + \sum_{n=N_1}^{N-1} z_n[v(t_n) - v(t_{n+1})],$$

which, because $0 \leq z_n \leq 1$ and $v(t_n) - v(t_{n+1}) \geq 0$ for all $n \geq N_1$, cannot be greater than v^*. Therefore, we have shown that arbitrageurs only play trigger-strategies, which, in turn, implies that the aggregate selling pressure s must be a non-decreasing function. On the other hand, all arbitrageurs sell in $[\underline{t}, \bar{t}]$, by virtue of which we have $s(t) = 0$ for $t < \underline{t}$ and $s(t) = \mu$ for $t > \bar{t}$. Hence,

$$F(t) = \frac{1}{\mu}s(t)$$

is a distribution function.

The distribution function F induces the equilibrium date of burst T. That is,

$$T(x) = \inf\{t|\mu F(t) \geq \kappa(x, t), t \geq 0\}. \tag{12}$$

Take any $0 \leq x_0 \leq 1$. It is obvious that $\mu F(T(x_0)) = \kappa(x_0, T(x_0))$. Also, for all $t < T(x_0)$, we have $T^{-1}(t) < x_0$ and, hence, $\mu F(t) < \kappa(x_0, t)$.

Part II: In this second part we show that there is no other equilibrium F fulfilling (12). Because any F is right-continuous, we know that

$$t_0 = \inf\{t|\mu F(t) \geq \kappa(x_0, t), t \geq 0\}$$

if, and only if, (i) $\mu F(t) < \kappa(x_0, t)$ for all $t < t_0$ and (ii) $\mu F(t_0) \geq \kappa(x_0, t_0)$. Therefore, any other equilibrium F must fulfill (ii) with strict inequality for some x_0. This means that it must have a jump at t_0. Now, let x_1 be the solution to $\kappa(x, t_0) = \mu F(t_0)$. We must have both $x_1 > x_0$ and $T(x_1) = T(x_0)$ (because (i) also implies that $\mu F(t) < \kappa(x_1, t)$ for all $t < t_0$), which contradicts Lemma 1.

Proof of Proposition 3. From the proof of Lemma 3 we know that $\bar{t} = [\pi - \arcsin(\mu/\overline{x})]\overline{x}$, where $\mu \leq \overline{x} \leq 1$. Therefore, $\bar{t}$ converges to $\pi/2$, *and not to zero*, as μ goes to 1.[9] Also, using (9) we have

$$v^* = e^{(g-r)[\pi - \arcsin(\mu/\overline{x})]\overline{x}}(1 - \overline{x}) + \overline{x} - c,$$

which converges to $1 - c$ as μ goes to 1. By (11) this implies that $\underline{t}$ converges to zero as μ goes to 1. By now, we have shown that the equilibrium support $[\underline{t}, \bar{t}]$ converges to the interval $[0, \pi/2]$ as μ goes to 1. In the limit, $T^{-1}(t)$ is thus given by (5) for all $t \leq \pi/2$. Lastly, v^* approaching $1-c$ implies that $T^{-1}(t)$ converges to 1 for all $t \leq \pi/2$. Plugging this and $\mu = 1$ into (6) gives the result because $\kappa(1, t) = \sin(t)$ for $t \leq \pi/2$ by (1).

[9] Recall that we are restricted to the principal branch of arcsin.