When financial imperfections are not the problem, but the solution

Maria Arvaniti *   Andrés Carvajal †

November 2, 2015
(*Preliminary draft*)

Abstract

The BP Deepwater Horizon oil spill of 2010 has focused considerable attention on the potential liability and the operating conduct of big oil companies. This paper suggests that, restricting the insurance opportunities of big oil companies in the asset markets creates incentives to internalize the welfare effects of such catastrophic events, leading to a Pareto improved allocation for society as a whole. We model a general equilibrium economy with uncertainty, where the probability of each state depends on the level of effort exerted by one agent. When this "effort" is costly and not contracted upon, every equilibrium is Pareto inefficient. We show that generically in the space of endowments, there is a Pareto-improving policy in the form of reallocation of existing assets. The results extend to economies with aggregate uncertainty and complete markets as well as to economies with uninsurable idiosyncratic risk. Hence, opposite to common wisdom, complete markets need not be constrained optimal in economies with this type of externalities.

Deepwater Horizon was an oil extraction rig owned by drilling company *Transocean*, which operates the largest fleet of deep-water offshore rigs in the world, and leased to *BP*, one of the world's leading energy companies, part of a conglomerate with interests that cover the whole range from oil and gas transportation fuels, to a diverse types of petrochemicals.[1] On April 20th, 2010, Deepwater Horizon was deployed some 66km off the coast of Louisiana, in the US portion of the Gulf of Mexico, when a concrete seal in the Macondo well burst as a consequence of a surge of natural gas. During the installation of that seal, contractor

* Umeå University; maria.arvaniti@econ.umu.se

† University of California–Davis and Fundação Getulio Vargas, EPGE; acarvajal@ucdavis.edu

[1] The use of which, in turn, extends to products as diverse as clothes and packaging.

Halliburton had used nitrogen to accelerate the "curing" of the concrete; this technique is known to weaken the pressure that this type of seal withstands.

The most immediate consequence of the explosion was the death of eleven members of the rig's crew, and the injuries caused to 17 more. After that, the oil spill that occurred during the next 87 days constitutes the largest one in the history of petroleum production, and the worst environmental catastrophe in US history. The damages caused to the ecosystem, to the local economy and to the population of the area remain largely unquantified, but some indicators are known.

During the sixth months following the explosion, some 8,000 dead animals were collected, with causes of death attributed directly to oil contamination. Later studies have shown that the number of local sea turtles becoming stranded onshore have multiplied five-fold, and that one half of the dolphins in the area exhibit serious health disorders known to be linked to oil exposure.[2] In the longer term, the number of seabird deaths attributed by scientists to the explosion ranges from 600,000 to 800,000.

Besides oil drilling, the economies of Louisiana, Mississippi and Alabama depend heavily on their fishing and tourism industries. Immediately after the explosion, the moratorium imposed on further drilling activities left an estimated 8,000–12,000 local workers unemployed. In response to the spill, government agencies deemed it necessary to close parts of the Gulf to all fishing activities, both commercial and recreational. At its peak, this ban covered 88,522 square miles, about one third of the US portion of the Gulf. Also, with 22% of the total news coverage in the US devoted to the incident during its first 100 days, the perception that seafood sourced in the gulf would be polluted is likely to have affected its demand, as it did the demand for tourism: according to a poll, in the first four months after the incident, almost one third of travellers to the coast of Louisiana had cancelled or postponed their visits because of the spill.

And while much more difficult to measure, especially in the short run, the incident had significant consequences for the lives of people local to the area. Again in the four months following the explosion, the clinical diagnoses of depression increased by about 25%, and more than half of the people responding a local survey reported to have been worried "almost constantly" about the spill.

In 2011, a report commissioned by the US government concluded that the explosion of the well could have been prevented, and that its causes could be traced directly to decisions made by BP, Halliburton and Transocean.

[2] This figure refers to a sub-area of the Gulf of Mexico, the Barataria Bay, in Louisiana.

By the time the well was successfully re-sealed, 4.9 million barrels of oil had been released to the waters of the Gulf. Three quarters of this oil could not be intercepted during the recovery efforts,[3] and while a significant part of this oil has naturally dispersed or evaporated, at least one quarter of it remained as residue, either at the water surface, floating as tar balls, washed ashore, or buried in sediment.

Shortly after the explosion, at the request of the US government BP placed in escrow a fund of $20bn to help address the financial losses caused by it. Later, it was also required to pay the $14bn cost of the clean-up efforts led by the US Coast Guard and other agencies, and on October 2015 the company reached a settlement agreement with the US Department of Justice for $20.8bn, which is meant to be comprehensive of all past and future damage at federal, state and local levels. One year after the incident, BP's market capitalization had decreased by about 23% (valued in Sterling) and during the first three quarters of 2010 the company stopped paying dividends to its shareholders. But by the fourth quarter of that year the dividends were reinstated at about one half the value of the last quarter of 2009, and between then and the second quarter of 2015 they have grown by 51%. Interestingly, while the company estimates the cost it has had to incur, up to the third quarter of 2015, in $55bn, its market capitalization has dropped only by some $32bn.

In January, 2011, American oil producer *Marathon* split its refinery activities from its exploration arm, after which its shares increased by 23%, gaining the company over $14bn in capital. The third-largest oil company in the country, *ConocoPhillips*, followed that strategy in May, 2012, and, even in the aftermath of the Deepwater Horizon spill, financial analysts stressed that doing the same would increase BP's market value by about $100bn.[4] It is apparent that the strategy of oil companies, in response to the lessons learnt during the Gulf of Mexico catastrophe, aims at isolating the revenue they make from other business from the potential losses caused by an exploration incident.[5] More in general, their intent appears to be to further diversify away the risk they bear from such incidents.

[3] The methods used to attempt interception, which included burning some of the oil at the surface and the application of chemical dispersers, are themselves likely to have a detrimental impact on eco-systems.

[4] See `www.bloomberg.com/news/2011-07-24/bp-breakup-worth-100-billion-to-jpmorgan.html`, which reports on a *JPMorgan*'s statement on this matter, and on similar positions held by analysts at *UBS* and *Bank of America*.

[5] This response would resemble the one that the industry had after the *Exxon Valdez* oil spill of March 24th, 1989. That incident occurred when the vessel of that name, through a series of human and operational errors, ran aground on Prince William Sound, Alaska. Before Deepwater Horizon, the spill that follows was the largest one ever, and its consequences affect the area even today: there still is oil residue affecting the Sound's beaches, waters and ecosystems. In 1993, the vessel's owner, Exxon, spun off all its shipping activities to a subsidiary.

This paper studies whether limiting the ability of an agent to insure, or in general to participate in financial trading to diversify their risks, can induce a Pareto improvement in the allocation of resources in the economy. That this may be the case is not obvious: the literature on financial economics shows that financial markets where not all the agents can trade freely fail to deliver Pareto efficiency. But this result is established under the assumption that no agent in the market causes an externality on others.

When Halliburton, Transocean and BP decided to accelerate the curing of the concrete on the seal of the Macondo Well, they increased the probability that an environmental catastrophe could occur in the Gulf. When they made their financial decisions, presumably they considered such increased probability, but only up the losses that they themselves would have to bear in the event of that catastrophe. The increased probability of losses for other agents is, in the language of Public Economics, an externality that the three companies did not internalize.

If the agent that causes an externality through the probability distribution of risks is limited in his access to financial markets, he certainly will respond by altering those probabilities in a way that is optimal to him. It seems easy to conclude that such policy can then be used to effect a Pareto improvement: in a partial equilibrium argument, this is immediately the case. The general equilibrium trade-off that we study, and which makes the conclusion less immediate and more interesting, is that in the presence of this kind of externalities, disturbing the functioning of perfect financial markets may be socially beneficial. The more obvious effect that a partial equilibrium argument ignores is the reaction of asset prices to the perturbation of the agent's financial portfolio. More subtle, but equally important, is the fact that, through market clearing, other agent's in the economy will have to change their portfolios too.

We study an economy where, initially, all agents participate in the trading of a complete set of financial assets. The only market failure in this economy is the externality that one agent imposes on the others via the probability distribution of their risks. We then show that, in a generic sense, everybody can be made better-off if the agent that causes the externality is not allowed to choose his portfolio of financial assets in an optimal manner, while everybody else continues to trade without constraints. Besides the restriction of that agent's financial decisions, lump-sum transfers are used to make sure that all agents are, indeed, made better-off. We first allow for a lump-sum transfer to the agent who imposes the externality. Being lump-sum, it is obvious that this transfer does *not* affect that agent's incentives, only his welfare. Still, it may be that for institutional reasons such transfer is not possible. Under the assumption that there is an external source of funding for the rest of the agents (which we call "relief aid"), we show that the Pareto improvement is still generically possible, even when the agent that imposes the externality is excluded from any lump-sum aid.

While the spirit of our results, and the techniques used by us, are similar to those in the literature on incomplete financial markets, it is worthwhile noticing that in our economy there is only one commodity per state, so that it is *not* via relative commodity prices that the Pareto improvement is induced by our financial policies.

1. Existing Literature

It is well known that any equilibrium allocation is Pareto efficient in competitive economies with complete financial markets. Constrained optimality is a weaker version of optimality that takes into account the financial structure and is appropriate for the case of incomplete markets. As Stiglitz [25], Greenwald and Stiglitz [16] and Geanakoplos and Polemarchakis [13] have argued, in a numéraire asset model with incomplete markets, generically, every equilibrium is constrained sub-optimal provided that there is more than one commodities and there is an upper bound in the number of individuals: reallocations of existing assets support superior allocations. Following Geanakoplos and Polemarchakis [13], Citanna, Kajii and Villanacci [11] show that equilibria are generically constrained inefficient even without an upper bound on the number of households. They also show that perfectly anticipated lump-sum transfers in a limited number of goods are typically effective.

The inefficiency of incomplete markets is the benchmark for the financial innovation literature. Hart [18] was the first to construct an example of a competitive economy where the equilibrium allocation with a single asset is Pareto dominated by one without no asset at all. Therefore, contrary to what one might have thought, the introduction of a new asset does not always improve upon welfare. As Cass and Cittana [8] and Elul [12] have argued, introducing a new asset in an economy with incomplete markets can be Pareto-improving or Pareto-impairing. Therefore, the choice of which assets to introduce has important welfare effects. This result has given rise to the literature on optimal security design. Prime examples of this strand of literature include Allen and Gale [1, 2], Chen [10] and Pesendorfer [22], where the main focus is the trade-off between profitable innovation and the cost of financial intermediation as a source of the incompleteness of the financial markets in equilibrium. Bisin [4] shows that in a environment with profit maximizing intermediaries and intermediation costs, markets are endogenously incomplete in equilibrium. Carvajal, Rostek and Weretka [7] show that competition among issuers of asset-backed securities who maximize asset value results in incomplete financial structures and inefficient risk sharing opportunities for the (prudent) investors, even if innovation is essentially costless.

This work is related also to the literature on moral hazard and financial innovation. In the moral hazard literature, the focus is on solving the principal-agent problem or in the case of multi-agent setup, on the reallocation of resources among individuals in order to solve trade-offs between incentives and risk sharing. Prime examples of this literature include Grossman and Hart [17] and Prescott and Townsend [24]. Helpman and Laffont [19] extend the moral hazard setup to a general equilibrium model with exogenously complete financial markets and non-exclusive transactions. A natural extension of this framework is to endogenize the financial design. In this direction, Lisboa [21] presents a general equilibrium model with individual and aggregate risk, moral hazard and a large number of households where insurance is supplied by a collection of profit maximizing firms that behave strategically. He shows that, given the assumptions of Bertrand competition and exclusivity, an equilibrium exists and is constrained optimal. On the other hand, Braido [6] presents a general equilibrium model as a two-stage game where agents act as producers, as consumers and as financial intermediaries with intermediation costs. Each individual is allowed to design a financial structure that consists of specifying securities pay-offs in each state and transaction constraints that restrict the participation of some agents in some markets, while allowing for non-exclusivity. He shows that an equilibrium exists and he offers examples and a reasoning as to why the equilibrium might be constrained inefficient and the markets incomplete. As he points out, apart from the financial technology and the strategic nature of financial innovation as standard sources of inefficiency, the main source would be the non-exclusivity of contracts. The inefficiency of equilibria in economies with moral hazard and non-exclusive contracts has been extensively studied by Helpman and Laffont [19], Arnott and Stiglitz [3], Bisin and Guaitoli [5] and Kahn and Mookherjee [20], where the main conclusion is that exclusivity is necessary for constrained efficiency of equilibria. It is important to point out the relevance of the work of Braido [6] for our model. To show how markets can be endogenously incomplete in equilibrium and how this might be optimal, he offers an example with two agents where only one of them is risk averse. This agent faces a production risk where the probability of the "good" state is increasing in the costly unobservable effort he exerts. In this setup, Braido shows that an incomplete financial structure, in the form of trading constraints, is Pareto superior to complete markets. Our framework is more general: all agents are risk-averse and face aggregate and/or idiosyncratic risk. In our setup, the agent who exerts effort creates an "externality." Moreover, by imposing trading constraints, we are limiting the insurance opportunities of the whole society and, since the rest of the agents are risk averse too, it is therefore more demanding to show that a Pareto improving policy exists.

The welfare improving policies that we analyze have been the subject of the restricted participation literature. As Polemarchakis and Siconolfi [23] point out, incomplete markets are just a special case of an asset market with restricted participation. In this setting, they prove the generic existence of competitive equilibria when agents face asymmetric linear constraints on their portfolio incomes. Cass, Siconolfi and Villanaci [9] extend the literature by accommodating a wide range of portfolio constraints, including any smooth, quasi-concave inequality constraint on households' portfolio holdings. Gori, Pireddu and Villanacci [15] focus on price-dependent borrowing restrictions. After proving existence of equilibrium, they show that equilibria associated with a sufficiently high number of strictly binding participation constraints in the financial markets can be Pareto improved upon by a local change in these constraints.

For the case of abstract economies with externalities and multiple commodities, Geanakoplos and Polemarchakis [14] show that competitive equilibrium is, generically, constrained inefficient: there exists an anonymous taxation policy that leaves all agents better off. While the concept of equilibrium used in that paper is the same as ours, neither the setting nor the type of externality are the same, and the mechanism through which the Pareto improvement is induced differs too.

2. An Externality Via Probabilities

Consider a two-period economy populated by $I + 1$ individuals, who are denoted by $i = 0, \ldots, I$. In period 0, agents receive an endowment of a single consumption good. In period 1, there is uncertainty regarding endowments: there are (only) two states of the world, which we denote by $s = 1, 2$, and agents receive an endowment e_s^i of the the consumption good in state s.[6]

Rather than being exogenous, the probabilities of the two states depend on some action of agent 0. In period 0, he has the choice of exerting costly *effort*, ε, which makes state 1 more likely. The cost of this effort is that it subtracts from the agent's consumption at date 0.

The preferences of the agents, who derive utility only from the single consumption good, are represented by

$$c_0^i + \pi(\varepsilon) \cdot u_1^i(c_1^i) + [1 - \pi(\varepsilon)] \cdot u_2^i(c_2^i), \tag{1}$$

[6] The results below will hold true if there are multiple commodities in the second period. In fact, results of the type we are studying are easier to argue in that case, only at the cost of heavier notation.

where $\pi(\varepsilon) \in (0, 1)$ is the probability of state 1. We allow for negativity of present consumption, c_0^i, so it is not necessary for us the specify date-0 endowments.

We assume that π is an increasing and concave function. Each state-dependent cardinal utility index u_s^i is assumed to be continuous, strictly increasing and strictly concave, and to satisfy standard Inada conditions.

The assumption that all individuals have preferences that are quasilinear in present consumption simplifies the mathematical arguments, but, of course, implies a loss of generality. The gained simplicity is that, since utility is transferable in this case, we can express our arguments in terms of aggregate *social welfare* functions.

We assume that $\varepsilon \in [\underline{\varepsilon}, \infty)$, for a lower bound on effort $\underline{\varepsilon}$. Now, in order to have interior solutions, we impose the following condition. For each state s, let $(\hat{c}_s^i)_{i=0}^I$ solve the following maximization problem:

$$\max_{(c^i)_{i=0}^I} \left\{ \sum_{i=0}^I u_s^i(c^i) : \sum_{i=0}^I c_s^i = \sum_{i=0}^I e_s^i \right\}. \tag{2}$$

The following assumptions will be technically useful later on:

Assumption **1** (Interiority). *At the consumption plan $\hat{c}^0$, agent 0 prefers state 1 to state 2, in the sense that $u_1^0(\hat{c}_1^0) > u_2^0(\hat{c}_2^0)$, and there exists a level of effort $\hat{\varepsilon} \in (\underline{\varepsilon}, \infty)$ such that*

$$\pi'(\hat{\varepsilon}) = \frac{1}{u_1^0(\hat{c}_1^0) - u_2^0(\hat{c}_2^0)},$$

Assumption **2** (Concavity). *Around the consumption plan $\hat{c}^0$ and the level of effort $\hat{\varepsilon}$, agent 0 has locally concave preferences: matrix*

$$\begin{pmatrix} \pi(\hat{\varepsilon}) \cdot \partial^2 u_1^0(\hat{c}_1^0) & 0 & \pi'(\hat{\varepsilon}) \cdot \partial u_1^0(\hat{c}_1^0) \\ 0 & [1 - \pi(\hat{\varepsilon})] \cdot \partial^2 u_2^0(\hat{c}_2^0) & -\pi'(\hat{\varepsilon}) \cdot \partial u_2^0(\hat{c}_2^0) \\ \pi'(\hat{\varepsilon}) \cdot \partial u_1^0(\hat{c}_1^0) & -\pi'(\hat{\varepsilon}) \cdot \partial u_2^0(\hat{c}_2^0) & \pi''(\hat{\varepsilon}) \cdot [u_1^0(\hat{c}_1^0) - u_2^0(\hat{c}_2^0)] \end{pmatrix} \tag{3}$$

is negative definite.

Assumption **3** (Heterogeneity). *At consumption allocation $\hat{c}$, agents other than 0 have heterogeneous preferences: for all $i, j \geqslant 1$,*

$$\partial^2 u_s^i(\hat{c}_s^i) \neq \partial^2 u_s^j(\hat{c}_s^j),$$

if $i \neq j$, for both $s = 1, 2$.

3. Competitive Equilibrium

Financial markets are assumed to be complete: there is an elementary (Arrow) security for each state, with asset s paying one unit of the consumption good in state s. Holdings of these securities are denoted by ϑ^i_s.

3.1. *The problem of agents* $i \geqslant 1$

All agents other than 0 have only one decision to make in period 0: they have to choose their holdings of securities, and therefore their consumption in that period and in both states in period 1. We assume that they take the prices of the securities and the probabilities of the states as given.

Letting q_1 and q_2 denote the prices of the securities, the problem of individual i is, simply,

$$\max_{\vartheta^i_1,\vartheta^i_2} \left\{-q_1 \cdot \vartheta^i_1 - q_2 \cdot \vartheta^i_2 + \pi(\varepsilon) \cdot u^i_1(e^i_1 + \vartheta^i_1) + [1 - \pi(\varepsilon)] \cdot u^i_2(e^i_2 + \vartheta^i_2)\right\}. \tag{4}$$

The first-order conditions of this problem are standard:[7]

$$q_1 = \pi(\varepsilon) \cdot \partial u^i_1(e^i_1 + \vartheta^i_1) \text{ and } q_2 = [1 - \pi(\varepsilon)] \cdot \partial u^i_2(e^i_2 + \vartheta^i_2). \tag{5}$$

These conditions are necessary and sufficient to characterize the solutions of Program (4).

3.2. *The problem of agent* $i = 0$

Agent 0 has an extra decision to make in period 0: apart from choosing his holdings of securities, he must choose his level of effort. His problem is, then:

$$\max_{\varepsilon,\vartheta^0_1,\vartheta^0_2} \left\{-\varepsilon - q_1 \cdot \vartheta^0_1 - q_2 \cdot \vartheta^0_2 + \pi(\varepsilon) \cdot u^0_1(e^0_1 + \vartheta^0_1) + [1 - \pi(\varepsilon)] \cdot u^0_2(e^0_2 + \vartheta^0_2)\right\}. \tag{6}$$

Assuming that this agent, too, takes asset prices as given, the following are the first-order conditions for any solution with an interior level of effort:

$$1 = \pi'(\varepsilon) \cdot [u^0_1(e^0_1 + \vartheta^0_1) - u^0_2(e^0_2 + \vartheta^0_2)], \tag{7}$$

while

$$q_1 = \pi(\varepsilon) \cdot \partial u^0_1(e^0_1 + \vartheta^0_1) \text{ and } q_2 = [1 - \pi(\varepsilon)] \cdot \partial u^0_2(e^0_2 + \vartheta^0_2). \tag{8}$$

[7] We use ∂u^i_s to denote the first derivative of function u^i_s; this is in lieu of the awkward notation $u^{i\prime}_s$.

Unfortunately, these conditions are only necessary, as we cannot guarantee the overall concavity of the individual's objective function. We will come back to this issue later, but, for the moment, note that the first-order condition with respect to effort, Eq. (7), implies that when agent 0 prefers state 1 to state 2 sufficiently, he is willing to exert effort to raise the likelihood of state 1. In his choice of effort, however, he does not internalize the effect of a more likely state 1 on the well-being of the society as a whole. Since we have made no assumptions on aggregate endowments and social welfare in each state, all agents other than agent 0 could be worse-off or better-off in state 1. It is this feature that reflects the non-alignment of interest across agents.

3.3. Nash-Walras Equilibrium

Competitive equilibrium is defined by individual rationality and market clearing requirements. We write competitive equilibria as a tuple $\{\bar{\vartheta}, \bar{\varepsilon}, \bar{q}\}$, where $\bar{\vartheta} = [(\bar{\vartheta}_s^i)_{s=1,2}]_{i=0}^I$ is the allocation of the two assets and $\bar{q} = (\bar{q}_1, \bar{q}_2)$ is the vector of asset prices, such that:

1. effort level $\bar{\varepsilon}$ and portfolio $(\bar{\vartheta}_s^0)_{s=1,2}$ solve Program (6) when the prices are $q = \bar{q}$;
2. for each $i \geqslant 1$, portfolio $(\bar{\vartheta}_s^i)_{s=1,2}$ solves Program (4) when the prices are $q = \bar{q}$ and the probability of state 1 is $\pi(\bar{\varepsilon})$; and
3. both of the securities markets clear: $\sum_{i=0}^I \bar{\vartheta}_s^i = 0$, for $s = 1, 2$.

For agents $i \geqslant 1$, individual rationality is characterized by the first-order conditions, Eq. (5). Importantly, we follow the spirit of Nash-Walras equilibrium in assuming that these agents take as given not only prices, but also the probabilities of the two states.

In the case of agent 0, in principle, the first-order conditions, Eqs. (7) and (8), are only necessary. Now, the combination of Eqs. (8) and (5) suffices to imply that, for each state of the world, the equilibrium allocation of consumption, given by $c_s^i = e_s^i + \bar{\vartheta}_s^i$, solves Program (2). Since the latter has a unique solution, by strict concavity of preferences, it follows that $\bar{\vartheta}_s^i = \hat{c}_s^i - e_s^i$. By Assumptions 1 and 2, it follows that Program (6) is concave in effort too, and hence that the first-order conditions are both necessary and sufficient. Note, then, that for agent 0, who causes an externality via his choice of effort, our assumption is that he takes prices as given, and considers the effects of effort on his own well-being only.

For later usage, let us define the function

$$\mathcal{F}(q,\vartheta,\varepsilon) = \begin{pmatrix} \pi(\varepsilon)\cdot \partial u_1^0(e_1^0+\vartheta_1^0) - q_1 \\ [1-\pi(\varepsilon)]\cdot \partial u_1^0(e_2^0+\vartheta_2^0) - q_2 \\ [u_1^0(e_1^0+\vartheta_1^0) - u_2^0(e_2^0+\vartheta_2^0)]\cdot \pi'(\varepsilon) - 1 \\ \pi(\varepsilon)\cdot \partial u_1^1(e_1^1+\vartheta_1^1) - q_1 \\ [1-\pi(\varepsilon)]\cdot \partial u_2^1(e_2^1+\vartheta_2^1) - q_2 \\ \vdots \\ \pi(\varepsilon)\cdot \partial u_1^I(e_1^I+\vartheta_1^I) - q_1 \\ [1-\pi(\varepsilon)]\cdot \partial u_2^I(e_2^I+\vartheta_2^I) - q_2 \\ \sum_{i=0}^I \vartheta_1^i \\ \sum_{i=0}^I \vartheta_2^i \end{pmatrix}.$$

Note that the roots of this function characterize the competitive equilibrium of the economy.

The following assumption is generically true, under Assumption 2.

Assumption 4 (Determinacy and trade). *At equilibrium, agent* 0 *participates in the market for elementary security* 1 *and the Jacobean of function* F *is invertible:* $\bar{\vartheta}_1^0 \neq 0$ *and matrix* $D\mathcal{F}(\bar{q},\bar{\vartheta},\bar{\varepsilon})$ *is non-singular.*

4. Pareto Efficiency

4.1. Definition

The definition of Pareto efficient allocation is as usual: a feasible allocation of agent 0's effort and the consumption of the unique commodity both at date 0 and in the two future states across all agents is efficient, if it is impossible to find an alternative, feasible allocation of these same variables that makes at least one agent better-off without making any other agent worse-off. Given that all individuals have quasilinear preferences, Pareto efficiency amounts to the choice of an allocation of consumption and a level of effort so as to solve the program

$$\max_{\varepsilon,(c_1^i,c_2^i)_{i=0}^I} \left\{ -\varepsilon + \sum_{i=0}^I \{\pi(\varepsilon)\cdot u_1^i(c_1^i) + [1-\pi(\varepsilon)]\cdot u_2^i(c_2^i)\} \mid \sum_{i=0}^I c_s^i = \sum_{i=0}^I e_s^i, s=1,2 \right\}. \quad (9)$$

Here, the date-0 consumption levels are left undetermined, but any allocation that exhausts the remaining aggregate endowment, net of ε, will be Pareto efficient.

The first-order conditions that characterize Pareto efficiency are, therefore, that

$$\pi'(\varepsilon) \cdot \sum_{i=0}^{I} [u_1^i(c_1^i) - u_2^i(c_2^i)] = 1; \tag{10}$$

and that, for each pair of agents $i, j = 0, \ldots, I$,

$$\partial u_s^i(c_s^i) = \partial u_s^j(c_s^j) \tag{11}$$

for each state $s = 1, 2$; as well as the feasibility condition.

4.2. Inefficiency of competitive equilibrium

Comparison of the first-order conditions of the individual, competitive choices of consumption, Eqs. (5) and (8), with the first-order condition defining efficiency of consumption plans, Eq. (11), shows that the allocations of consumption prescribed by the competitive equilibrium are efficient in the sense that the marginal rates of substitutions would be equalized across agents in each state.

On the other hand, from the first-order conditions for the choice of effort, Eqs. (7) and (10), it is immediate that the level of effort implied by the competitive equilibrium solution is generically not Pareto optimal: while agent 0 takes into account only the effects on his own welfare, the social planner considers the effects on social welfare when choosing the optimal level of effort.

5. Constrained inefficiency of competitive equilibrium

5.1. Two definitions

We have shown that competitive equilibria need not yield the Pareto efficient level of effort. This result says that if a planner could choose the effort exerted by agent 0, he would choose a different level, and then would reallocate date-0 consumption to make sure that all agents, including 0 himself, are made better-off.

As is usual in the General Equilibrium literature, the latter observation does not mean that a social planner who faces constraints in terms of the policies he can apply would indeed be able to effect a welfare improving policy. Here, we consider the case in which the planner is constrained in the sense that he can *only* distort the asset holdings of agent 0, but cannot directly choose the effort level that the agent exerts. We also allow the planner to effect lump-sum transfers of revenue across all agents, perhaps including 0 himself. If such a policy exists that leaves everybody better-off, we shall say that the competitive equilibrium is *constrained inefficient*. Whether a lump sum transfer to agent 0 is required determines the how strong the definition is.

Formally, we say that an allocation (ε, c), where $c = [(c^i_s)_{s=0,1,2}]^I_{i=0}$,[8] is *weakly constrained inefficient* if there exist an alternative level of effort, $\tilde{\varepsilon}$; asset prices, $\tilde{q}$; a profile of asset holdings, $\tilde{\vartheta}$; and a profile of date-0 lump-sum transfers, $(\tau^i)^I_{i=0}$, such that:

1. $\tilde{\varepsilon}$, the level of effort of agent 0, solves
$$\max_{\varepsilon} \left\{-\varepsilon + \pi(\varepsilon) \cdot u^0_1(e^0_1 + \tilde{\vartheta}^0_1) + [1 - \pi(\varepsilon)] \cdot u^0_2(e^0_2 + \tilde{\vartheta}^0_2)\right\}; \tag{12}$$

2. for each $i \geqslant 1$, portfolio $(\tilde{\vartheta}^i_s)_{s=1,2}$ solves Program (4) when the prices are $q = \tilde{q}$ and the probability of state 1 is $\pi(\tilde{\varepsilon})$;

3. both of the securities markets clear: $\sum^I_{i=0} \tilde{\vartheta}^i_s = 0$, for $s = 1, 2$;

4. the profile of lump-sum transfers is balanced: $\sum^I_{i=0} \tau^i = 0$;

5. agent 0 is better-off, in that
$$-\tilde{\varepsilon} - \tilde{q}_1 \cdot \tilde{\vartheta}^0_1 - \tilde{q}_2 \cdot \tilde{\vartheta}^0_2 + \tau^0 + \pi(\tilde{\varepsilon}) \cdot u^0_1(e^0_1 + \tilde{\vartheta}^0_1) + [1 - \pi(\tilde{\varepsilon})] \cdot u^0_2(e^0_2 + \tilde{\vartheta}^0_2)$$
is higher than
$$-\varepsilon + c^0_0 + \pi(\varepsilon) \cdot u^0_1(c^0_1) + [1 - \pi(\varepsilon)] \cdot u^0_2(c^0_2);$$

6. every agent $i \geqslant 1$ is better-off, in that
$$-\tilde{q}_1 \cdot \tilde{\vartheta}^i_1 - \tilde{q}_2 \cdot \tilde{\vartheta}^i_2 + \tau^i + \pi(\tilde{\varepsilon}) \cdot u^i_1(e^i_1 + \tilde{\vartheta}^i_1) + [1 - \pi(\tilde{\varepsilon})] \cdot u^i_2(e^i_2 + \tilde{\vartheta}^i_2)$$
is higher than
$$c^i_0 + \pi(\varepsilon) \cdot u^i_1(c^i_1) + [1 - \pi(\varepsilon)] \cdot u^i_2(c^i_2).$$

Intuitively, the allocation is constrained inefficient if a planner can effect a Pareto improvement by forcing agent 0 out of the financial markets: instead of letting him choose optimal holdings of the two assets, a portfolio $\tilde{\vartheta}^0$ is allocated to him. Then competitive assets markets open for *every other* agent in the economy, and assets prices are determined

[8] The *feasibility* condition that $\sum^I_{i=0} c^i_s = \sum^I_{i=0} e^i_s$, for both $s = 1, 2$, will hold throughout our analysis. Also, note that in the following analysis c^0_0 is taken to be *gross* of effort.

endogenously. Agent 0's choices are limited to his effort level. Yet, at the allocation induced (endogenously) by the policy, every agent is strictly better-off.

What makes this definition weak is that we are allowing for agent 0 to receive a lump-sum transfer beyond the resulting price value of the portfolio imposed on him. It is important to note that this transfer does *not* affect the agent's incentives in choosing how much effort to exert.[9] Yet, for institutional reasons it may be impossible for the planner to effect such transfer. We shall say that the allocation is constrained inefficient *in the strong sense*, if such Pareto improvement is possible even when τ^0 is required to be null.

5.2. Genericity of weak constrained inefficiency

Fix competitive equilibrium, $\{\bar{\vartheta}, \bar{\varepsilon}, \bar{q}\}$. Our first goal is to show that, generically on the date-1 endowments of individuals, the allocation $(\bar{\varepsilon}, \bar{c})$, where $\bar{c}^i_s = e^i_s + \bar{\vartheta}^i_s$ for $s = 1, 2$, is constrained inefficient in the weak sense.

Since the weak definition of constrained inefficiency allows for transfers across all agents, we can again use the fact that preferences are quasilinear to write an expression for social welfare,

$$W = -\varepsilon + \pi(\varepsilon) \cdot \sum_{i=0}^{I} u^i_1(c^i_1) + [1 - \pi(\varepsilon)] \cdot \sum_{i=0}^{I} u^i_2(c^i_2). \tag{13}$$

Recall that the problem arises from the inefficient level of effort chosen by agent 0 when maximizing his utility in the competitive setting. We consider an exogenous perturbation in the holdings of securities of agent 0. The idea is to restrict his insurance opportunities to make him more vulnerable to the risks associated with each state. Such perturbation, $(d\vartheta^0_1, d\vartheta^0_2)$, around the competitive equilibrium values induces changes in all other endogenous variables: from Eq. (7), it follows that the effort exerted by agent 0 will change; this will induce different consumption and investment decisions by all agents $i \geqslant 1$; and, in order to guarantee market clearing, the prices of assets will need to accommodate too. If, starting from the equilibrium allocation we can find that there exists a perturbation such that $dW > 0$, we can conclude that the allocation is constrained inefficient in the weak sense: the higher value of the social welfare function W implies the existence of the required profile of lump-sum transfers after which every agent in the economy is made strictly better-off.

The total change on the social welfare function, dW, is given by the aggregate of three effects:

[9] So, the planner is *not* bribing him to exert more effort.

1. the direct effect through the change in the level of effort exerted by agent 0, $-d\varepsilon$;

2. an indirect effect through the different likelihood of the two states that is induced by that change,

$$\sum_{i=0}^{I}[u_1^i(\bar{c}_1^i) - u_2^i(\bar{c}_2^i)] \cdot \pi'(\bar{\varepsilon}) \cdot d\varepsilon;$$

3. and an indirect effect through the changes in the consumption plans of all individuals, in response to the different likelihoods, the change in prices, and, for agent 0, his exogenously perturbed portfolio:

$$\pi(\bar{\varepsilon}) \cdot \sum_{i=0}^{I} \partial u_1^i(\bar{c}_1^i) \cdot dc_1^i + [1 - \pi(\bar{\varepsilon})] \cdot \sum_{i=0}^{I} \partial u_2^i(\bar{c}_2^i) \cdot dc_2^i.$$

Since the only source of inefficiency in our model is the level of effort, the third effect vanishes. Indeed, using Eqs. (5) and (8), the total effect can be written simply as

$$dW = \sum_{i=1}^{I}[u_1^i(\bar{c}_1^i) - u_2^i(\bar{c}_2^i)] \cdot \pi'(\bar{\varepsilon}) \cdot d\varepsilon + \bar{q}_1 \cdot \sum_{i=0}^{I} dc_1^i + \bar{q}_2 \cdot \sum_{i=0}^{I} dc_2^i.$$

By the feasibility constraint, $\sum_{i=0}^{I} dc_s^i = 0$, so that, finally,

$$dW = \sum_{i=1}^{I}[u_1^i(\bar{c}_1^i) - u_2^i(\bar{c}_2^i)] \cdot \pi'(\bar{\varepsilon}) \cdot d\varepsilon. \tag{14}$$

Now, we can solve for $d\varepsilon$ by differentiating the first-order condition of agent 0 with respect to effort, Eq. (7), to obtain[10]

$$d\varepsilon = \frac{\pi'(\bar{\varepsilon}) \cdot \left[\partial u_2^0(e_2^0 + \bar{\vartheta}_2^0) \cdot dc_2^0 - \partial u_1^0(e_1^0 + \bar{\vartheta}_1^0) \cdot dc_1^0\right]}{\pi''(\bar{\varepsilon}) \cdot [u_1^0(e_1^0 + \bar{\vartheta}_1^0) - u_2^0(e_2^0 + \bar{\vartheta}_2^0)]}. \tag{15}$$

The direction of the Pareto improving policy depends on the sign of expression

$$\sum_{i=1}^{I}[u_1^i(\bar{c}_1^i) - u_2^i(\bar{c}_2^i)].$$

Positive values imply that the society, excluding agent 0, is better-off in state 1 than in state 2. As a result, $d\varepsilon$ must be positive: the competitive level of effort is too low and the

[10] Note that the interiority condition imposed above, namely Assumption 1, implies that

$$u_1^0(e_1^0 + \bar{\vartheta}_1^0) - u_2^0(e_2^0 + \bar{\vartheta}_2^0) > 0,$$

since $e_s^0 + \bar{\vartheta}_s^0 = \hat{c}_s^0$.

Pareto improving policy involves inducing agent 0 to increase the level of effort chosen at equilibrium. Looking at Eq. (15), this can be achieved with $d\vartheta_2^0 < 0$ and $d\vartheta_1^0 > 0$. That is, in order to induce agent 0 to exert more effort in equilibrium, a planner would like to restrict his insurance opportunities in a way that he is better-off in the state associated with higher effort, namely state 1, and worse-off in the other state. The agent who causes the externality is, therefore, "forced" to internalize the externality through considerations of his own welfare.[11]

It is important to note that, generically in the space of endowments,

$$\sum_{i=1}^{I} [u_1^i(\bar{c}_1^i) - u_2^i(\bar{c}_2^i)] \neq 0, \tag{16}$$

so that there almost always is a Pareto improving policy. The proof of this result is shown in the appendix. This analysis implies, then, our first main result.

THEOREM **1.** *If the economy satisfies Assumptions 1 and 2, then, except on a negligible set of individual endowments, the competitive equilibrium allocation is weakly constrained inefficient.*

6. CATASTROPHE, RELIEF AID AND GENERICITY OF STRONG CONSTRAINED INEFFICIENCY

If the policy that aims at effecting a Pareto improvement is restricted to not include agent 0 in the profile of date-0 transfers, we can no longer use the social welfare function W. For agent 0, the policy needs to increase

$$U = -\varepsilon - q_1 \cdot \tilde{\vartheta}_1^0 - q_2 \cdot \tilde{\vartheta}_2^0 + \pi(\varepsilon) \cdot u_1^0(e_1^0 + \tilde{\vartheta}_1^0) + [1 - \pi(\varepsilon)] \cdot u_2^0(e_2^0 + \tilde{\vartheta}_2^0), \tag{17}$$

while, simultaneously, increasing

$$-q_1 \cdot \sum_{i=1}^{I} \vartheta_1^i - q_2 \cdot \sum_{i=1}^{I} \vartheta_2^i + \pi(\varepsilon) \cdot \sum_{i=1}^{I} u_1^i(e_1^i + \vartheta_1^i) + [1 - \pi(\varepsilon)] \cdot \sum_{i=1}^{I} u_2^i(e_2^i + \vartheta_2^i).$$

This would suffice, as the higher value of the latter sub-aggregate implies the existence of the required sub-profile of lump-sum transfers, $(\tau^i)_{i=1}^I$, after which all the agents $i \geqslant 1$ are made strictly better-off.

[11] Of course, in the case where $\sum_{i=1}^{I}[u_1^i(\bar{c}_1^i) - u_2^i(\bar{c}_2^i)] < 0$, the society excluding agent 0 is better-off in state 2, so that the competitive level of effort is inefficiently high: in this case, Pareto optimality prescribes $d\varepsilon < 0$ which can be achieved with $d\vartheta_2^0 > 0$ and $d\vartheta_1^0 < 0$. The intuition is, again, that one would like to make agent 0 better-off in the state associated with lower effort and worse-off in the other state so that it is optimal for him to choose a lower level of effort than before.

The following two assumptions will allow us to prove that, in a generic sense, the competitive equilibrium allocation is strongly constrained inefficient.

Assumption 5 (Catastrophe). *At the consumption allocation* $(\hat{c}^i)_{i=0}^I$*, the whole society prefers state 1 to state 2, in the sense that*

$$\mu^0 = u_1^0(\hat{c}_1^0) - u_2^0(\hat{c}_2^0) > 0 \tag{18}$$

and

$$\mu^{\neg 0} = \sum_{i=1}^{I} [u_1^i(\hat{c}_1^i) - u_2^i(\hat{c}_2^i)] > 0. \tag{19}$$

Assumption 6 (Relief aid). *There exists an external source of funding that aids agents* $i = 1, \ldots, I$ *in their purchases of the elementary security for state 2. This fund covers a total of* $\rho > 0$ *units of the asset.*

We assume that this aid takes the form of lump-sum transfers (of the correct value at equilibrium), so that they have no impact on the characterization of equilibrium we have used. Under the latter assumption, we can write the aggregate utility of agents $i = 1, \ldots, I$ as

$$V = -q_1 \cdot \sum_{i=1}^{I} \vartheta_1^i - q_2 \cdot \left(\sum_{i=1}^{I} \vartheta_2^i - \rho \right) + \pi(\varepsilon) \cdot \sum_{i=1}^{I} u_1^i(e_1^i + \vartheta_1^i) + [1 - \pi(\varepsilon)] \cdot \sum_{i=1}^{I} u_2^i(e_2^i + \vartheta_2^i). \tag{20}$$

6.1. Local subspaces of functions

Under the quasi-linearity assumption, the equilibrium allocation is $\hat{c}$ and the equilibrium level of effort is $\hat{\varepsilon}$. In order to perform genericity analysis on the spaces of utilities and the probability function, we parameterize a local subspace of functions as follows. Given some $\beta > 0$, let $b : \mathbb{R} \to [0, 1]$ be a C^1 function such that

$$b(x) = \begin{cases} 0, & \text{if } |x| > \beta; \\ 1, & \text{if } |x| < \beta/2. \end{cases}$$

We refer to this function as a *bump*. Also for $\bar{\delta} > 0$, consider the «perturbed» mappings

$$(\varepsilon, \delta_0) \mapsto \pi(\varepsilon) + b(\varepsilon - \bar{\varepsilon}) \cdot \delta_0 \cdot (\varepsilon - \bar{\varepsilon})^2,$$

and, for each $i \geqslant 1$ and $s = 1, 2$,

$$(c_s^i, \delta_s^i) \mapsto u_s^i(c_s^i) + b(c_s^i - \bar{c}_s^i) \cdot \delta_s^i \cdot (c_s^i - \bar{c}_s^i)^2.$$

For simplicity of notation, we will write these perturbed functions as $\pi(\cdot;\delta_0)$ and $u^i_s(\cdot;\delta^i_s)$.

Note that if β and $\bar{\delta}$ are small enough, these mappings are increasing and concave, so long as $|\delta_0| < \bar{\delta}$ and $|\delta^i_s| < \bar{\delta}$ for all $i = 1, \ldots, I$ and $s = 1, 2$. This implies that we can use each of these parameters to perturb the corresponding function on an one-dimensional open neighborhood of the original function. For technical reasons, we restrict attention to the open subset

$$\Delta = \{\delta \in (-\bar{\delta}, \bar{\delta})^5 \mid \partial^2 u^1_1(\hat{c}^1_1) + \delta^1_1 \neq \partial^2 u^2_1(\hat{c}^2_1) + \delta^2_1 \text{ and } \partial^2 u^1_2(\hat{c}^1_2) + \delta^1_2 \neq \partial^2 u^2_2(\hat{c}^2_2) + \delta^2_2\}. \quad (21)$$

Under Assumption 3, it is immediate that $0 \in \Delta$.

The following lemma is immediate.

Lemma 1 (Invariance of equilibrium to perturbations). *Perturbations to the probability function and preferences of agents* $i = 1, \ldots, I$ *do not affect the equilibrium: since* $\mathcal{F}(\bar{q}, \bar{\vartheta}, \bar{\varepsilon}) = 0$, *tuple* $(\bar{q}, \bar{\vartheta}, \bar{\varepsilon})$ *continues to be an equilibrium when the probability function is* $\pi(\cdot;\delta_0)$ *and preferences are* $u^i_s(\cdot;\delta^i_s)$, *as long as* $|\delta_0| < \bar{\delta}$ *and* $|\delta^i_s| < \bar{\delta}$ *for all* $i = 1, \ldots, I$ *and* $s = 1, 2$.

The lemma exploits one key property of our construction: that the perturbations do *not* affect the first derivatives of the perturbed functions at the equilibrium values. What they do, however, is to affect their second derivatives, which is the second key property — one that we will use below.

6.2. *A characterization of strong constrained inefficiency*

In order to make the concept of strong constrained inefficient easier to analyse, define the following function, where, for simplicity, we assume that $I = 2$,

$$(q, \vartheta, \varepsilon, \delta) \mapsto \begin{pmatrix} U \\ V \\ [u^0_1(e^0_1 + \vartheta^0_1) - u^0_2(e^0_2 + \vartheta^0_2)] \cdot \pi'(\varepsilon;\delta_0) - 1 \\ \pi(\varepsilon;\delta_0) \cdot \partial u^1_1(e^1_1 + \vartheta^1_1;\delta^1_1) - q_1 \\ [1 - \pi(\varepsilon;\delta_0)] \cdot \partial u^1_2(e^1_2 + \vartheta^1_2;\delta^1_2) - q_2 \\ \pi(\varepsilon;\delta_0) \cdot \partial u^2_1(e^2_1 + \vartheta^2_1;\delta^2_1) - q_1 \\ [1 - \pi(\varepsilon;\delta_0)] \cdot \partial u^2_2(e^2_2 + \vartheta^2_2;\delta^2_2) - q_2 \\ \sum_{i=0}^{I} \vartheta^i_1 \\ \sum_{i=0}^{I} \vartheta^i_2 \end{pmatrix}, \quad (22)$$

where $\mathcal{U}$ and $\mathcal{V}$ are as in Eqs. (17) and (20), respectively, with $\pi(\cdot;\delta_0)$ and $u_s^i(\cdot;\delta_s^i)$ instead of $\pi(\cdot)$ and $u_s^i(\cdot)$.[12]

Denoting this mapping by $\mathcal{H}(q,\vartheta,\varepsilon;\delta)$, the following lemma follows as an implication of the definition of strong constrained inefficiency, and of the construction of functions $\mathcal{U}$ and $\mathcal{V}$.

LEMMA 2. *If $\{\bar\vartheta,\bar\varepsilon,\bar q\}$ is a competitive equilibrium and the partial Jacobean of $\mathcal{H}$ with respect to $(q,\vartheta,\varepsilon)$ has full row rank, then the equilibrium allocation $(\bar\varepsilon,\bar c)$, where $\bar c_s^i = e_s^i + \bar\vartheta_s^i$ for $s=1,2$, is constrained inefficient in the strong sense.*

Proof. If the Jacobean

$$D_{q,\vartheta,\varepsilon}\mathcal{H}(\bar q,\bar\vartheta,\bar\varepsilon;\delta)$$

has full row rank, by the inverse function theorem it follows that $\mathcal{H}(\cdot;\delta)$ maps a neighbourhood of $(\bar q,\bar\vartheta,\bar\varepsilon)$ *onto* a neighbourhood of $\mathcal{H}(\bar q,\bar\vartheta,\bar\varepsilon;\delta)$. It then follows that, for a small enough $d>0$, we can find $(\tilde q,\tilde\vartheta,\tilde\varepsilon)$ such that

$$\mathcal{H}(\tilde q,\tilde\vartheta,\tilde\varepsilon;\delta) = \mathcal{H}(\bar q,\bar\vartheta,\bar\varepsilon;\delta) + (d,d,0,\dots,0)^{\mathsf T}.$$

At equilibrium, by definition, the third to last entries of $\mathcal{H}(\bar q,\bar\vartheta,\bar\varepsilon;\delta)$ equal 0. Substituting, this means that

$$\begin{pmatrix} \mathcal{U}(\tilde q,\tilde\varepsilon,\tilde\vartheta;\delta) \\ \mathcal{V}(\tilde q,\tilde\varepsilon,\tilde\vartheta;\delta) \\ \left[u_1^0(e_1^0+\tilde\vartheta_1^0) - u_2^0(e_2^0+\tilde\vartheta_2^0)\right]\cdot\pi'(\tilde\varepsilon;\delta_0) - 1 \\ \pi(\varepsilon;\delta_0)\cdot\partial u_1^1(e_1^1+\tilde\vartheta_1^1;\delta_1^1) - \tilde q_1 \\ [1-\pi(\varepsilon;\delta_0)]\cdot\partial u_2^1(e_2^1+\tilde\vartheta_2^1;\delta_2^1) - \tilde q_2 \\ \pi(\varepsilon;\delta_0)\cdot\partial u_1^2(e_1^2+\tilde\vartheta_1^2;\delta_1^2) - \tilde q_1 \\ [1-\pi(\varepsilon;\delta_0)]\cdot\partial u_2^2(e_2^2+\tilde\vartheta_2^2;\delta_2^2) - \tilde q_2 \\ \sum_{i=0}^{I}\tilde\vartheta_1^i \\ \sum_{i=0}^{I}\tilde\vartheta_2^i \end{pmatrix} = \begin{pmatrix} \mathcal{U}(\bar q,\bar\varepsilon,\bar\vartheta;\delta)+d \\ \mathcal{V}(\bar q,\bar\varepsilon,\bar\vartheta;\delta)+d \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first entry of this equality and the fact that $d>0$ imply that agent 0 is made better-off, which is the fifth requirement of the definition of weak constrained inefficiency. The second entry implies that the aggregate $\mathcal{V}$ of utilities is higher too, which in turn implies that every *other* agent can be made better-off by an appropriate choice of lump-sum transfers,

[12] When writing a function, the notation $f(x;\alpha)$ is used to emphasize that x is the function's argument and α is just a parameter. Hopefully, this makes it clear that $\partial f(x;\alpha)$ refers to the derivative of the function with respect to argument x, when the parameter takes the value α.

hence guaranteeing the sixth requirement in the definition (without violating the fourth requirement). Noting that the third entry implies the first-order condition of Program (12), the first requirement is also satisfied. The second requirement follows from the fourth to the ante-penultimate entries, as each successive pair of them implies the first-order conditions of Program (4), for agents $i = 1, \dots, I$. Finally, the last two entries imply the third requirement of the definition. □

6.3. *Genericity of strong constrained inefficiency*

The characterization of constrained inefficiency by Lemma 2 seemingly allows us to prove a second generic result.

CLAIM **1.** *If the economy satisfies Assumptions 1 to 6, then, on an open and dense set of individual endowments, probability functions and individual preferences, the competitive equilibrium allocation is strongly constrained inefficient.*

Fixing all preferences and endowments such that the assumptions hold true, the challenge is to show that mapping

$$(q, \vartheta, \varepsilon, \delta, \gamma) \mapsto \begin{pmatrix} \mathcal{F}(q, \vartheta, \varepsilon; \delta) \\ D_{q,\vartheta,\varepsilon}\mathcal{H}(q, \vartheta, \varepsilon; \delta)^{\mathsf{T}}\gamma \\ \frac{1}{2}(\gamma \cdot \gamma - 1) \end{pmatrix},$$

where $\gamma \in \mathbb{R}^{2+2(I+1)+1}$, is transverse to 0. Let $\mathcal{M}$ denote the previous mapping.

Note that the entries of vector γ can be named according to the rows of the Jacobean matrix $D_{q,\vartheta,\varepsilon}\mathcal{H}$. Recalling the definition of mapping $\mathcal{H}$, these rows are:

1. the utility level of agent 0, U;
2. the aggregate utility of all other agents, V;
3. the first-order condition with respect to effort, in the maximization of agent 0;
4. for each i other than 0 and each s, the first-order condition with respect to ϑ^i_s, in the maximization of agent i; and
5. for each s, the market clearing condition of the corresponding elementary security.

It is then convenient to use a mnemonic to denote the entries of the vector, as follows:

$$\gamma = (\gamma_U, \gamma_V, \gamma^0, \gamma^1_1, \gamma^1_2, \gamma^2_1, \gamma^2_2, \gamma_1, \gamma_2).$$

Lemma 3 (Simplifying the system). *Suppose that* $\mathcal{M}(q,\vartheta,\varepsilon,\delta,\gamma)=0$. *Then,*

1. $(q,\vartheta,\varepsilon)=(\bar{q},\bar{\vartheta},\bar{\varepsilon})$;
2. *the second component of* $\mathcal{M}$, *namely* $D_{q,\vartheta,\varepsilon}\mathcal{H}(\bar{q},\bar{\vartheta},\bar{\varepsilon};\delta)^{\mathsf{T}}\gamma$, *simplifies to*

$$\begin{pmatrix} \pi'\cdot\sum_{i=1}^{2}[(u_1^i-u_2^i)\cdot\gamma_V+(\partial u_1^i\cdot\gamma_1^i-\partial u_2^i\cdot\gamma_2^i)]+(\pi''+\delta_0)\cdot(u_1^0-u_2^0)\cdot\gamma^0 \\ \pi'\cdot\partial u_1^0\cdot\gamma^0+\gamma_1 \\ -\pi'\cdot\partial u_2^0\cdot\gamma^0+\gamma_2 \\ (\partial^2u_1^1+\delta_1^1)\cdot\gamma_1^1+\gamma_1 \\ (\partial^2u_2^1+\delta_2^1)\cdot\gamma_2^1+\gamma_2 \\ (\partial^2u_1^2+\delta_1^2)\cdot\gamma_1^2+\gamma_1 \\ (\partial^2u_2^2+\delta_2^2)\cdot\gamma_2^2+\gamma_2 \\ -\vartheta_1^0\cdot(\gamma_U-\gamma_V)-\gamma_1^1-\gamma_1^2 \\ -\vartheta_2^0\cdot(\gamma_U-\gamma_V)+\rho\cdot\gamma_V-\gamma_2^1-\gamma_1^2 \end{pmatrix} \qquad (*)$$

3. $\gamma_1\neq 0$ *and* $\gamma_2\neq 0$;
4. $\gamma^0\neq 0$, *and for both* $i=1,2$ *and both* $s=1,2$, $\gamma_s^i\neq 0$; *and*
5. *either* $\gamma_1^1\neq\gamma_1^2$ *or* $\gamma_2^1\neq\gamma_2^2$.

Proof. The first statement follows immediately from Lemma 1, as $\mathcal{M}=0$ implies that $\mathcal{F}=0$. For the second statement, it follows by direct computation that $D_{q,\vartheta,\varepsilon}\mathcal{H}(\bar{q},\bar{\vartheta},\bar{\varepsilon};\delta)^{\mathsf{T}}\gamma$ equals the sum of expression $(*)$ and

$$\begin{pmatrix} [-1+\pi'\cdot(u_1^0-u_2^0)]\cdot\gamma_U \\ (-q_1+\pi\cdot\partial u_1^0)\cdot\gamma_U \\ [-q_2+(1-\pi)\cdot\partial u_2^0]\cdot\gamma_U \\ (-q_1+\pi\cdot\partial u_1^1)\cdot\gamma_V \\ [-q_2+(1-\pi)\cdot\partial u_2^1]\cdot\gamma_V \\ (-q_1+\pi\cdot\partial u_1^2)\cdot\gamma_V \\ [-q_2+(1-\pi)\cdot\partial u_2^2]\cdot\gamma_V \\ 0 \\ 0 \end{pmatrix}.$$

At $(\bar{q},\bar{\vartheta},\bar{\varepsilon})$, this latter vector vanishes.

For the third statement, suppose, by way of contradiction, that $\gamma_1=0$. It follows from the second equation of the system $D_{q,\vartheta,\varepsilon}\mathcal{H}(\bar{q},\bar{\vartheta},\bar{\varepsilon};\delta)^{\mathsf{T}}\gamma=0$ that $\gamma^0=0$. In the third equation, this implies that $\gamma_2=0$ too. The fourth to seventh equations then imply that

$$\gamma_1^1=\gamma_2^1=\gamma_1^2=\gamma_2^2=0.$$

By Assumption 4, the previous-to-last equation implies that $\gamma_u = \gamma_V$. Then, Assumption 6 implies, via the last equation, that $\gamma_V = 0$. Summing up, $\gamma = 0$, which is impossible since $\gamma \cdot \gamma = 1$, as $\mathcal{M} = 0$.

For the fourth statement, note that if $\gamma^0 = 0$, the second equation implies that $\gamma_1 = 0$, which we just showed to be impossible. The same occurs if $\gamma_s^i = 0$: then, from the fourth to seventh equations it would follow that $\gamma_s = 0$, contradicting the third statement.

For the last statement, suppose to the contrary that $\gamma_1^1 = \gamma_1^2$ and $\gamma_2^1 = \gamma_2^2$. The fourth and fifth equations imply, given Assumption 6, that $\gamma_u = \gamma_V = 0$. From the last four equations we then have that

$$\partial^2 u_1^1 + \delta_1^1 = \partial^2 u_1^2 + \delta_1^2 \text{ and } \partial^2 u_2^1 + \delta_2^1 = \partial^2 u_2^2 + \delta_2^2,$$

which is impossible by the definition of the space Δ of perturbations, Eq. (21). □

LEMMA 4 (Transversality). *Mapping $\mathcal{M}$ is transverse to 0.*

Proof. Suppose that $\mathcal{M} = 0$. It follows from Lemma 1 that its Jacobean can be written as

$$\begin{pmatrix} D_{q,\vartheta,\varepsilon}\mathcal{F} & 0 \\ M & \Omega \end{pmatrix},$$

where Ω is the partial Jacobean of mapping

$$(q, \vartheta, \varepsilon, \delta, \gamma) \mapsto \begin{pmatrix} D_{q,\vartheta,\varepsilon}\mathcal{H}(q, \vartheta, \varepsilon; \delta)^\top \gamma \\ \frac{1}{2}(\gamma \cdot \gamma - 1) \end{pmatrix}$$

with respect to (δ, γ).[13]

Then, given Assumption 4, all we need to show is that matrix Ω has full row rank. To make it easier to see that this is indeed the case, it is convenient to re-organize this matrix. Note that the first rows of this matrix correspond to the entries of the product $D_{q,\vartheta,\varepsilon}\mathcal{H}^\top \gamma$.[14] By construction, these rows correspond to the arguments with respect to which mapping $\mathcal{H}$ has been differentiated, which we have been writing in the order

$$(\varepsilon, \vartheta_1^0, \vartheta_2^0, \vartheta_1^1, \vartheta_2^1, \vartheta_1^2, \vartheta_2^2, q_1, q_2).$$

It is now convenient, in fact, to write the last two rows of the matrix in the fourth and fifth positions, which amounts to taking the derivatives of $\mathcal{H}$ in the order

$$(\varepsilon, \vartheta_1^0, \vartheta_2^0, q_1, q_2, \vartheta_1^1, \vartheta_2^1, \vartheta_1^2, \vartheta_2^2).$$

[13] Matrix M is the partial Jacobean of this same mapping with respect to $(q, \vartheta, \varepsilon)$. We need not concern ourselves with its computation.

[14] These rows are followed by only one other: the derivatives of the last row of $\mathcal{M}$.

Of course, the rank of Ω is not affected by this operation. As for its columns, it will also be convenient to write them in an unusual order:

$$(\delta^0, \gamma_1, \gamma_2, \gamma_u, \gamma_V, \delta_1^1, \delta_2^1, \delta_1^2, \delta_2^2, \gamma_1^1, \gamma_2^1, \gamma_1^2, \gamma_2^2).$$

One remaining argument, γ^0, will not be necessary for our differentiations.

Permuted in this way, when $\mathcal{M} = 0$ matrix Ω reads as

$$\begin{pmatrix}
\mu^0\gamma_0 & 0 & 0 & 0 & \pi'\mu^{\neg 0} & 0 & 0 & 0 & 0 & \pi'\partial u_1^1 & -\pi'\partial u_2^1 & \pi'\partial u_1^2 & -\pi'\partial u_2^2 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\theta_1^0 & \theta_1^0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & -\theta_2^0 & \theta_2^0 + \rho & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & \pi\gamma_1^1 & 0 & 0 & 0 & \pi h_1^1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & (1-\pi)\gamma_2^1 & 0 & 0 & 0 & (1-\pi)h_2^1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \pi\gamma_1^2 & 0 & 0 & 0 & \pi h_1^2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1-\pi)\gamma_2^1 & 0 & 0 & 0 & (1-\pi)h_2^2 \\
0 & \gamma_1 & \gamma_2 & \gamma_u & \gamma_V & 0 & 0 & 0 & 0 & \gamma_1^1 & \gamma_2^1 & \gamma_1^2 & \gamma_2^2
\end{pmatrix},$$

where μ^0 and $\mu^{\neg 0}$ come from Eqs. (18) and (19), respectively, and h_s^i is used to denote $\partial^2 u_s^i + \delta_s^i$, for brevity.

Consider first the leading principal minor of order 5, namely the matrix

$$\begin{pmatrix}
\mu^0 \cdot \gamma_0 & 0 & 0 & 0 & \pi' \cdot \mu^{\neg 0} \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -\theta_1^0 & \theta_1^0 \\
0 & 0 & 0 & -\theta_2^0 & \theta_2^0 + \rho
\end{pmatrix}.$$

Under Assumptions 5 and 6, this matrix is non-singular, thanks to the fact that $\gamma^0 \neq 0$, as per Lemma 3. Now, add the next four columns and rows. This adds, the columns

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\pi \cdot \gamma_1^1 & 0 & 0 & 0 \\
0 & (1-\pi) \cdot \gamma_2^1 & 0 & 0 \\
0 & 0 & \pi \cdot \gamma_1^2 & 0 \\
0 & 0 & 0 & (1-\pi) \cdot \gamma_2^1
\end{pmatrix}.$$

Given that for both $i = 1, 2$ and both $s = 1, 2$, $\gamma_s^i \neq 0$, again as per Lemma 3, it follows that this whole 9×9 leading principal minor is non-singular.

It only remains to show that when we add the last row of the matrix and the four remaining columns, the whole matrix maintains its full row rank. Given the last statement in Lemma 3, we can assume, with no loss of generality, that $\gamma_1^1 \neq \gamma_1^2$. Now, perform the following operations:

1. add the columns corresponding to γ_1^1;
2. subtract the column corresponding to γ_1^2;
3. subtract h_1^1/γ_1^1 times the column corresponding to δ_1^1;
4. add h_1^2/γ_1^2 times the column corresponding to δ_1^2; and
5. subtract
$$\frac{\pi' \cdot (\partial u_1^1 - \partial u_1^2)}{(u_1^0 - u_2^0) \cdot \gamma_0}$$
times the column corresponding to δ_0, which can be done thanks to Assumption 5 and the fourth statement in Lemma 3.

Note that the result of these operations is vector

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \gamma_1^1 - \gamma_1^2 \end{pmatrix}$$

Since $\gamma_1^1 \neq \gamma_1^2$, the last entry of this resulting vector is non-zero. This implies that the matrix, as a whole, can also span the last dimension of its co-domain, and, given the previous results, that it has full row rank. $\square$

We are now ready to prove that, generically on endowments, preferences and the probability function, the competitive equilibrium allocation is costrained inefficient in the strong sense too, under the extra assumptions.

Proof of Claim 1. Since $M \pitchfork 0$, by the transversality theorem we will have that $\mathcal{M}(\cdot, \delta) \pitchfork 0$, generically on $\delta \in \Delta$. Now, $\mathcal{M}(\cdot, \delta)$ has

$$3 + 2I + 2 + 2 + 2(I + 1) + 1 + 1$$

entries, and only

$$2 + 2(I + 1) + 1 + 2 + 2(I + 1) + 1$$

arguments. It follows that $D_{q,\vartheta,\varepsilon,\gamma}\mathcal{M}$ cannot have full row rank, and hence it must be that, generically on $\delta \in \Delta$, $\mathcal{M}(q, \vartheta, \varepsilon, \gamma, \delta) \neq 0$.

Now, if that is the case, it is immediate that whenever

$$\mathcal{F}(q, \vartheta, \varepsilon, e,) = 0 \text{ and } D_{q,\vartheta,\varepsilon}\mathcal{H}(\bar{q}, \bar{\vartheta}, \bar{\varepsilon}, e)^{\mathsf{T}}\gamma = 0,$$

we also have that $\gamma = 0$. This implies, by construction, that when $\mathcal{F}(q, \vartheta, \varepsilon, e, u, \pi) = 0$, matrix $D_{q,\vartheta,\varepsilon}\mathcal{H}(\bar{q}, \bar{\vartheta}, \bar{\varepsilon}, e)$ has full row rank. By Lemma 2, it follows that the equilibrium allocation is constrained inefficient, in the strong sense, generically on $\delta \in \Delta$

This result implies the claim, as long as we endow the space of preference and probability functions with a metrizable topology: for any given initial functions, any open neighborhood of these functions will intersect the lower-dimensional subspace Δ, since $0 \in \Delta$. Since the equilibrium allocation is strongly constrained inefficient generically on Δ, we can find an array of functions in the intersection of the open neighborhood and Δ where this is the case. This implies denseness, as needed. □

7. More sophisticated behaviour by agent 0

Our analysis so far has assumed that all agents, including 0, take assets prices as given. This is the application to our context of the standard definition of Nash-Walras equilibrium from Public Economics. In Financial Economics, on the other hand, when an agent has the ability to affect the probability distribution of shocks in the economy, it is usually considered that he recognizes the effect that her decisions *with respect to those probabilities* affect the willingness to pay of other agents for the existing financial assets and, therefore, the equilibrium level of assets prices. In a sense, the point is that the kind of externality that agent 0 induces in our economy is *too salient* for him to take everybody else's actions as given.

We now recognize this possibility and study whether our previous results are robust to this more sophisticated behaviour by agent 0. In order to do this, we need to decompose asset prices into the part of them that depends on the probability distribution and the part that is determined by trade, given the probabilities. Effectively, it is as if decisions were made sequentially in the first period of: individual 0 first chooses the level of effort he is to commit, and then, taken that level of effort as given, he and all the other agents trade financial assets in a competitive manner.

Luckily, our formulation allows us to nest these two decisions in just one problem for agent 0. In order to do this, we first introduce two functions that are of common use in financial economics. For each state s, and each level of trading of the corresponding elementary security for that state by agent 0, ϑ_s^0, let $(\tilde{c}_s^i)_{i=1}^I$ denote the solution to the following maximization problem:

$$\max_{(c^i)_{i=1}^I} \left\{ \sum_{i=1}^I u_s^i(c^i) : \sum_{i=1}^I c_s^i = \sum_{i=0}^I e_s^i - \vartheta_s^0 \right\}. \tag{23}$$

Obviously, the solution to this problem depends on ϑ_s^0, and so we can define

$$\kappa_s(\vartheta_s^0) = \partial u_s^1(\tilde{c}_s^1),$$

a function to which we will refer as the (ex-post) *pricing kernel* for state s.

The utility of this function is that If $\{\bar{\vartheta}, \bar{\varepsilon}, \bar{q}\}$ is a competitive equilibrium (in the sense defined before), it follows from Eq. (5), that the equilibrium prices decompose into the product of the pricing kernel for each state and its corresponding probability:

$$\begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix} = \begin{pmatrix} \pi(\bar{\varepsilon}) \cdot \kappa_1(\bar{\vartheta}_1^0) \\ [1 - \pi(\bar{\varepsilon})] \cdot \kappa_2(\bar{\vartheta}_2^0) \end{pmatrix}$$

Since we want to maintain the assumption that all agents are competitive in the financial markets, we will now consider the situation in which agent 0 recognizes the direct effect that his choice of effort has on the vector of asset prices, via the probabilities, but acts as if his choice of portfolio did not affect the pricing kernels. The behaviour of all other agents in the economy remains unchanged, as well as the market clearing condition. For the sake of definiteness, we refer to this situation as *Stackelberg-Walras equilibrium*, defined as follows: it is a tuple $\{\bar{\vartheta}, \bar{\varepsilon}, \bar{q}\}$, where $\bar{\vartheta} = [(\bar{\vartheta}_s^i)_{s=1,2}]_{i=0}^I$ is the allocation of the two assets and $\bar{q} = (\bar{q}_1, \bar{q}_2)$ is the vector of asset prices, such that:

1. effort level $\bar{\varepsilon}$ and portfolio $(\bar{\vartheta}^0_s)_{s=1,2}$ solve the following Program:

$$\max_{\varepsilon,\vartheta^0_1,\vartheta^0_2} \{-\varepsilon - q_1(\varepsilon;\bar{\vartheta}^0_1)\vartheta^0_1 - q_2(\varepsilon;\bar{\vartheta}^0_2)\vartheta^0_2 + \pi(\varepsilon)u^0_1(e^0_1+\vartheta^0_1) + [1-\pi(\varepsilon)]u^0_2(e^0_2+\vartheta^0_2)\} \tag{24}$$

   where $q_1(\varepsilon;\vartheta^0_1) = \pi(\varepsilon)\cdot\kappa_1(\vartheta^0_1)$ and $-q_2(\varepsilon;\vartheta^0_2) = [1-\pi(\varepsilon)]\cdot\kappa_2(\vartheta^0_2)$;

2. for each $i \geqslant 1$, portfolio $(\bar{\vartheta}^i_s)_{s=1,2}$ solves Program (4) when the prices are $q = \bar{q}$ and the probability of state 1 is $\pi(\bar{\varepsilon})$; and

3. both of the securities markets clear: $\sum_{i=0}^{I} \bar{\vartheta}^i_s = 0$, for $s = 1, 2$.

Importantly, as agent 0 recognizes the effect of his effort decision on prices, Eq. (7) is no longer a valid first-order condition of his problem; instead, now his behaviour is characterized by the requirement that

$$1 = \pi'(\bar{\varepsilon})\cdot\{-[\kappa_1(\bar{\vartheta}^0_1) - \kappa_2(\bar{\vartheta}^0_2)] + [u^0_1(e^0_1+\bar{\vartheta}^0_1) - u^0_2(e^0_2+\bar{\vartheta}^0_2)]\} \tag{25}$$

along with Eqs. (8) evaluated at $(\bar{\varepsilon}, \bar{\vartheta}^0_1, \bar{\vartheta}^0_2)$.

7.2. Weak constrained inefficiency

As before, if we fix a Stackelberg-Walras equilibrium, $\{\bar{\vartheta}, \bar{\varepsilon}, \bar{q}\}$, our goal is again to show that the allocation $(\bar{\varepsilon}, \bar{c})$, where $\bar{c}^i_s = e^i_s + \bar{\vartheta}^i_s$, is constrained inefficient in the weak sense, generically on date-1 endowments.

With the different first-order condition with respect to effort, Eq. (25), the results of §5.2 imply that, instead of Eq. (14), we obtain

$$dW = \left\{[\kappa_1(\bar{\vartheta}^0_1)\cdot\bar{\vartheta}^0_1 - \kappa_2(\bar{\vartheta}^0_2)\cdot\bar{\vartheta}^0_2] + \textstyle\sum_{i=1}^{I}[u^i_1(\bar{c}^i_1) - u^i_2(\bar{c}^i_2)]\right\}\cdot\pi'(\bar{\varepsilon})\cdot d\varepsilon \tag{26}$$

while

$$d\varepsilon = \frac{\pi'(\bar{\varepsilon})\cdot\left[\bar{\vartheta}^0_2\cdot\partial^2 u^1_2(e^1_2+\bar{\vartheta}^1_2)\cdot d\vartheta^0_2 - \bar{\vartheta}^0_1\cdot\partial^2 u^1_1(e^1_1+\bar{\vartheta}^1_1)\cdot d\vartheta^0_1\right]}{\pi''(\bar{\varepsilon})\cdot\left\{[\kappa_1(\bar{\vartheta}^0_1)\cdot\bar{\vartheta}^0_1 - \kappa_2(\bar{\vartheta}^0_2)\cdot\bar{\vartheta}^0_2] + [u^0_1(e^0_1+\bar{\vartheta}^0_1) - u^0_2(e^0_2+\bar{\vartheta}^0_2)]\right\}}. \tag{27}$$

Generically on date-1 endowments, a policy can induce $d\varepsilon$ such that $dW > 0$.

8. Uninsurable idiosyncratic risk

As an alternative framework, and in order to show that these results extend to economies with idiosyncratic risk, we now study a model where the agents are subject to uninsurable idiosyncratic shocks. For the sake of brevity, we restrict attention to the weaker definition of constrained inefficiency.

Individuals are of different types, $i = 0, \dots, I$, and within each type there is a continuum of individuals of mass m^i. For simplicity, we assume that $m^0 = 1$.

Individuals of different types differ in their period-1 preferences and in their endowments, but they face the same idiosyncratic shocks. For simplicity, let us assume that there are only three personal states, denoted by $s = 1, 2, 3$. In $s = 1$ there is no shock in the endowment of the consumption good, while in $s = 2, 3$ there is a positive and negative shock, respectively, of size z. In period 1, a fraction $\pi(\varepsilon)$ of the individuals find themselves in state 1, while equal fractions of size $\frac{1}{2}[1-\pi(\varepsilon)]$ find themselves in states 2 and 3. However, there is no aggregate uncertainty: the aggregate endowment of the economy in period 1 is $\sum_{i=0}^{I} m^i \cdot \bar{e}^i$, where $\bar{e}^i$ is the endowment of individuals of type i in state 1, where there is no shock.

Once again, the probabilities of each personal state depend on the aggregate effort that agents of type 0 will choose to exert. In particular, while the expected value of the endowment remains unchanged with effort, higher effort increases the probability of observing no shock and decreases the probability of positive and negative shocks. A lower effort, in other words, induces a mean-preserving spread in the distributions of the agents' wealth. Therefore, a risk averse agent would prefer a distribution that second-order stochastically dominates, and this is why he would decide to exert effort.

8.1. Competitive equilibrium

Suppose there is only a risk-less asset that can be traded: it pays one unit of the consumption good at date 1. Holdings of the asset are b^i and its price is q.

8.1.1. The problem of agents of type $i \geqslant 1$

Once again, all agents of types other than 0 have to choose their holdings of the riskless bond, and therefore the consumption in period 0 and in every personal state in period 1. The problem that an agent of type $i \geqslant 1$ faces is to choose b^i so as to maximize

$$-q \cdot b^i + \pi(\varepsilon) \cdot u^i(e^i + b^i) + \frac{1}{2} \cdot [1-\pi(\varepsilon)] \cdot [u^i(e^i + z + b^i) + u^i(e^i - z + b^i)].$$

These agents take the price of the riskless bond and the probabilities of the personal states as given. The first-order condition of this problem is, then, that

$$q = \pi(\varepsilon) \cdot \partial u^i(e^i + b^i) + \frac{1}{2} \cdot [1 - \pi(\varepsilon)] \left[\partial u^i(e^i + z + b^i) + \partial u^i(e^i - z + b^i)\right]. \qquad (28)$$

8.1.2. The problem of agents of type 0

On the other hand, each agent of type 0 has to maximize

$$-\varepsilon - q \cdot b^0 + \pi(\varepsilon) \cdot u^0(e^0 + b^0) + \frac{1}{2} \cdot [1 - \pi(\varepsilon)] \cdot [u^0(e^0 + z + b^0) + u^0(e^0 - z + b^0)]$$

by his choice of savings, b^0, and effort, ε.

It is imporant to note that, although we have assumed that the probability of each state depends on the aggregate effort exerted by all agents of type 0, when an agent j of type 0 solves his maximization problem, he sees the probability of each state as depending only on his own level of effort.[15] In equilibrium, this does not matter as all agents of type 0 choose the same level of effort.

Here, the first-order conditions are that[16]

$$1 = \pi'(\varepsilon) \cdot \left[(u^0(e^0 + b^0) - \frac{1}{2} \cdot u^0(e^0 + z + b^0) - \frac{1}{2} \cdot u^0(e^0 - z + b^0)\right] \qquad (29)$$

and

$$q = \pi(\varepsilon) \cdot \partial u^0(e^0 + b^0_j) + \frac{1}{2} \cdot [1 - \pi(\varepsilon)] \cdot \left[\partial u^0(e^0 + z + b^0) + \partial u^0(e^0 - z + b^0)\right]. \qquad (30)$$

8.1.3. Nash-Walras equilibrium

As before, the first-order conditions of individual rationality and the market clearing requirement characterize competitive equilibrium. We denote competitive equilibria by $\{\bar{b}, \bar{\varepsilon}, \bar{q}\}$, where $\bar{b} = (\bar{b}^i)_{i=0}^I$ is a profile of savings. These values solve the first-order conditions, Eqs. (28), (29) and (30), as well as the equality $\sum_{i=0}^I m^i \cdot b^i = 0$.

[15] That is, strictly speaking: each agent j of type 0 chooses a level of effort ε_j, considering the probability $\pi(\varepsilon_j)$; agents of types $i \geqslant 1$, on the other hand, take as given the probability $\pi(\int \varepsilon_j \, dj)$.

[16] As with the case of complete markets, we can make interiority assumptions so that we do not have to look for boundary solutions.

Now, consider a policy intervention that perturbs by db^0 the holdings of the riskless bond of all agents of type 0.The welfare effects of such policy around the competitive equilibrium point, dW are, as before, the sum of:

1. the direct loss due to a different level of effort, $-d\varepsilon$;

2. the indirect effect due to the change in probabilities,

$$\sum_{i=0}^{I} m^i \cdot \left\{u^i(\bar{c}_1^i) - \frac{1}{2} \cdot \left[u^i(\bar{c}_2^i) + u^i(\bar{c}_3^i)\right]\right\} \cdot \pi'(\bar{\varepsilon}) \cdot d\varepsilon,$$

   where $\bar{c}_s^i$ represents the equilibrium consumption of agents of type i in state s; and

3. the indirect effect due to the reallocation of savings,

$$\pi(\bar{\varepsilon}) \cdot \sum_{i=0}^{I} m^i \cdot \partial u^i(\bar{c}_1^i) \cdot dc_1^i + \frac{1}{2} \cdot [1 - \pi(\bar{\varepsilon})] \cdot \sum_{i=0}^{I} m^i \cdot \left[\partial u^i(\bar{c}_2^i) \cdot dc_2^i + \partial u^i(\bar{c}_3^i) dc_3^i\right].$$

As before, taking into account the first-order conditions of the agents at the equilibrium point and the market clearing conditions, this expression simplifies to

$$dW = \sum_{i=1}^{I} m^i \cdot \left\{u^i(\bar{c}_1^i) - \frac{1}{2} \cdot \left[u^i(\bar{c}_2^i) + u^i(\bar{c}_3^i)\right]\right\} \cdot \pi'(\bar{\varepsilon}) \cdot d\varepsilon, \tag{31}$$

where we can substitute

$$d\varepsilon = -\frac{\pi'(\bar{\varepsilon})}{\pi''(\bar{\varepsilon})} \cdot \frac{\partial u^0(\bar{c}_1^0) \cdot dc_1^0 - \frac{1}{2} \cdot [\partial u^0(\bar{c}_2^0) \cdot dc_2^0 + \partial u^0(\bar{c}_3^0) \cdot dc_3^0]}{u^0(\bar{c}_1^0) - \frac{1}{2} \cdot [u^0(\bar{c}_2^0) + u^0(\bar{c}_3^0)]}. \tag{32}$$

By strong concavity,

$$\sum_{i=1}^{I} m^i \cdot u^i(e^i + \bar{b}^i) - \frac{1}{2} \cdot \sum_{i=1}^{I} m^i \cdot u^i(e^i + z + \bar{b}^i) - \frac{1}{2} \cdot \sum_{i=1}^{I} m^i \cdot u^i(e^i - z + \bar{b}^i) > 0,$$

which implies that the competitive level of effort is inefficiently low: an increase in the level of effort would be welfare improving for the whole society. Then, Eq. (15) implies that the direction of the perturbation of the bond holdings of agent 0 that implements a higher level of effort depends on the sign of expression

$$\partial u^0(e^0 + \bar{b}^0) \cdot dc_1^0 - \frac{1}{2} \cdot \left[\partial u^0(e^0 + z + \bar{b}^0) \cdot dc_2^0 + \partial u^0(e^0 - z + \bar{b}^0) \cdot dc_3^0\right].$$

If we now assume that agents of type 0 are *prudent*, we conclude that a perturbation $db^0 < 0$ induces an improvement in social welfare: if these agents save below the equilibrium level, with convex marginal utility they will exert a higher level of effort since by doing so, they make the future look less "volatile". This reduction in volatility makes the aggregate welfare higher, in spite of the fact that a higher effort subtracts from the aggregate welfare functions and even though it implies an imperfect operation of the financial market.

Appendix A

Lemma 5. *Eq.* (16) *is true at competitive equilibrium, generically in the space of endowments.*

Proof. At competitive equilibrium, the allocation of commodities satisfies the first-order conditions of Pareto efficiency *with respect to the allocation of commodities* — compare Eqs. (5) and (8) with Eq. (11). Thus, we start by defining the following function

$$\mathcal{K}(c_1, c_2, \lambda_1, \lambda_2, e_1, e_2) = \begin{pmatrix} \partial u_1^0(c_1^0) - \lambda_1 \\ \partial u_1^1(c_1^1) - \lambda_1 \\ \vdots \\ \partial u_1^I(c_1^I) - \lambda_1 \\ \sum_{i=0}^I (e_1^i - c_1^i) \\ \sum_{i=1}^I [u_1^i(c_1^i) - u_2^i(c_2^i)] \\ \partial u_2^0(c_2^0) - \lambda_2 \\ \partial u_2^1(c_2^1) - \lambda_2 \\ \vdots \\ \partial u_2^I(c_2^I) - \lambda_2 \\ \sum_{i=0}^I (e_2^i - c_2^i) \end{pmatrix} \tag{33}$$

which includes those conditions, the resource constraints for each state in period 1 and the utility sub-aggregate $\sum_{i=1}^I [u_1^i(c_1^i) - u_2^i(c_2^i)]$.[17] Then, at any competitive equilibrium of the economy, the 4-tuple $(c_1, c_2, \lambda_1, \lambda_2)$ is such that all entries of $\mathcal{K}$ other than the one corresponding to $\sum_{i=1}^I [u_1^i(c_1^i) - u_2^i(c_2^i)]$, are equal to 0. With arguments in the order

$$(c_1^0, \dots, c_1^I, e_1^0, \lambda_1, c_2^0, \dots, c_2^I, \lambda_2, e_1^1, \dots, e_1^I, e_2^0, \dots, e_2^I),$$

[17] We denote by λ_1 and λ_2 the Lagrange multipliers associated with the resource constraints in states 1 and 2 respectively. These can be constructed by taking the ratio of the price of each elementary security and the probability of the corresponding state, at equilibrium.

the Jacobian at the Pareto efficient allocation at each state is

$$\begin{pmatrix}
\partial^2 u_1^0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \partial^2 u_1^1 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \partial^2 u_1^I & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
-1 & -1 & \cdots & -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & \partial u_1^1 & \cdots & \partial u_1^I & 0 & 0 & 0 & \partial u_2^1 & \cdots & \partial u_2^I & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \partial^2 u_2^0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \partial^2 u_2^1 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \partial^2 u_2^I & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1
\end{pmatrix}. \tag{34}$$

We now argue that this $[2(I+1)+3] \times [4(I+1)+2]$ matrix has full rank, in the following four steps:

1. Consider the submatrix without the last $2I+1$ columns, which we denote with $\mathcal{V}$, and partition this submatrix as

$$\left(\begin{array}{cccccc|ccccc}
\partial^2 u_1^0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \partial^2 u_1^1 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \partial^2 u_1^I & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & -1 & \cdots & -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \partial u_1^1 & \cdots & \partial u_1^I & 0 & 0 & 0 & \partial u_2^1 & \cdots & \partial u_2^I & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & \partial^2 u_2^0 & 0 & \cdots & 0 & -1 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \partial^2 u_2^1 & \cdots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \partial^2 u_2^I & -1 \\
0 & 0 & \cdots & 0 & 0 & 0 & -1 & -1 & \cdots & -1 & 0
\end{array}\right). \tag{35}$$

2. We now show that the top-left $(I+3) \times (I+3)$ submatrix, which we denote $\mathcal{V}_{11}$, has full row rank. First notice that, by strong concavity, the submatrix without the last row has full row rank. Then, for $\mathcal{V}_{11}$ to have full row rank, it suffices to show that there exists a column vector $\alpha \in \mathbb{R}^{I+3}$ such that $\mathcal{V}_{11}\alpha = (0, \beta)^\top$ for some scalar $\beta \neq 0$.

Let

$$\alpha = \begin{pmatrix} \partial^2 u_1^0(\hat{c}_1^0)^{-1} \\ \partial^2 u_1^1(\hat{c}_1^1)^{-1} \\ \vdots \\ \partial^2 u_1^I(\hat{c}_1^I)^{-1} \\ \sum_{i=0}^{I} \partial^2 u_1^i(\hat{c}_1^i)^{-1} \\ 1 \end{pmatrix}.$$

Then, $\mathcal{V}_{11}\alpha$ is 0 everywhere, except from the last row where it is

$$\beta = \sum_{i=1}^{I} \frac{\partial u_1^i(\hat{c}_1^i)}{\partial^2 u_1^i(\hat{c}_1^i)} \neq 0.$$

3. Noting that $\mathcal{V}_{11}$ is invertible, $\mathcal{V}_{21}$ is a zero matrix and $\mathcal{V}_{22}$ has full row rank (by strong concavity), and since

$$|\mathcal{V}| = |\mathcal{V}_{11}||\mathcal{V}_{22} - \mathcal{V}_{21}\mathcal{V}_{11}^{-1}\mathcal{V}_{12}|,$$

we conclude that $\mathcal{V}$ has full row rank.

4. Finally, adding the last $2I+1$ columns the row rank of the matrix will not change, so $D\mathcal{K}$ has full row rank.

Now, since $D\mathcal{K}$ has full row rank, we conclude that $\mathcal{K} \pitchfork 0$. Then, the set of endowments at which $\mathcal{K}(\cdot, e_1, e_2) \pitchfork 0$ has full measure. Since $D\mathcal{K}(\cdot, e_1, e_2)$ has one fewer column than it has rows, $\mathcal{K}(\cdot, e_1, e_2) \pitchfork 0$ implies that, whenever all conditions for competitive equilibrium are true, Eq. (16) is also true. □

References

[1] Allen, F., and Gale, D. Optimal security design. *Review of Financial Studies 1*, 3 (1988), 229–263.

[2] Allen, F., and Gale, D. Arbitrage, short sales, and financial innovation. *Econometrica* (1991), 1041–1068.

[3] Arnott, R., and Stiglitz, J. Equilibrium in competitive insurance markets with moral hazard. Tech. rep., National Bureau of Economic Research, 1991.

[4] Bisin, A. General equilibrium with endogenously incomplete financial markets. *Journal of Economic Theory 82*, 1 (1998), 19–45.

[5] Bisin, A., and Guaitoli, D. Moral hazard and nonexclusive contracts. *The RAND Journal of Economics* (2004), 306–328.

[6] Braido, L. General equilibrium with endogenous securities and moral hazard. *Economic Theory 26*, 1 (2005), 85–101.

[7] Carvajal, A., Rostek, M., and Weretka, M. Competition in financial innovation. *Econometrica 80*, 5 (2012), 1895–1936.

[8] Cass, D., and Citanna, A. Pareto improving financial innovation in incomplete markets. *Economic Theory 11*, 3 (1998), 467–494.

[9] Cass, D., Siconolfi, P., and Villanacci, A. Generic regularity of competitive equilibria with restricted participation. *Journal of Mathematical Economics 36*, 1 (2001), 61–76.

[10] Chen, Z. Financial innovation and arbitrage pricing in frictional economies. *Journal of Economic Theory 65*, 1 (1995), 117–135.

[11] Citanna, A., Kajii, A., and Villanacci, A. Constrained suboptimality in incomplete markets: a general approach and two applications. *Economic Theory 11*, 3 (1998), 495–521.

[12] Elul, R. Welfare effects of financial innovation in incomplete markets economies with several consumption goods. *Journal of Economic Theory 65*, 1 (1995), 43–78.

[13] Geanakoplos, J., and Polemarchakis, H. Existence, regularity and constrained suboptimality of competitive equilibrium allocations when the asset market is incomplete. *Uncertainty, information and communication: essays in honor of Kenneth Arrow 3* (1986), 65–96.

[14] Geanakoplos, J., and Polemarchakis, H. M. Pareto improving taxes. *Journal of Mathematical Economics 44*, 7 (2008), 682–696.

[15] Gori, M., Pireddu, M., and Villanacci, A. Regularity and pareto improving on financial equilibria with price-dependent borrowing restrictions. *Research in Economics* (2012).

[16] Greenwald, B., and Stiglitz, J. Externalities in economies with imperfect information and incomplete markets. *The Quarterly Journal of Economics 101*, 2 (1986), 229–264.

[17] Grossman, S., and Hart, O. An analysis of the principal-agent problem. *Econometrica* (1983), 7–45.

[18] Hart, O. On the optimality of equilibrium when the market structure is incomplete. *Journal of Economic Theory 11*, 3 (1975), 418–443.

[19] Helpman, E., and Laffont, J.-J. On moral hazard in general equilibrium theory. *Journal of Economic Theory 10*, 1 (1975), 8–23.

[20] Kahn, C., and Mookherjee, D. Competition and incentives with nonexclusive contracts. *The RAND Journal of Economics* (1998), 443–465.

[21] Lisboa, M. Moral hazard and general equilibrium in large economies. *Economic Theory 18*, 3 (2001), 555–575

[22] Pesendorfer, W. Financial innovation in a general equilibrium model. *Journal of Economic Theory 65*, 1 (1995), 79–116.

[23] Polemarchakis, H., and Siconolfi, P. Generic existence of competitive equilibria with restricted participation. *Journal of Mathematical Economics 28*, 3 (1997), 289–311.

[24] Prescott, E., and Townsend, R. Pareto optima and competitive equilibria with adverse selection and moral hazard. *Econometrica* (1984), 21–45.

[25] Stiglitz, J. The inefficiency of the stock market equilibrium. *The Review of Economic Studies 49*, 2 (1982), 241–261.